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TABLE OF CONTENTS

Foundations, Theory of Sets, Logic 237
Algebra 240
Combinatorial Analysis 240
Linear Algebra 241
Polynomials 243
Partial Order, Lattices 243
Fields, Rings 244
Algebras 246
Groups and Generalizations 248
Homological Algebra 249
Theory of Numbers 250
General Theory of Numbers 250
Analytic Theory of Numbers 252
Theory of Algebraic Numbers 253
Geometry of Numbers 253
Analysis 253
Functions of Real Variables 253
Measure, Integration 255
Functions of Complex Variables 257
Geometric Analysis 260
Harmonic Functions, Convex Functions . . . 261
Special Functions 263
Sequences, Series, Summability 264
Approximations, Orthogonal Functions . . . 267
Trigonometric Series and Integrals 268
Integral Transforms 270
Ordinary Differential Equations 271
Partial Differential Equations 279
Difference Equations, Functional Equations . 288
Integral and Integrodifferential Equations . . 289
Calculus of Variations 289
Topological Algebraic Structures 289
Topological Groups 289
Lie Groups and Algebras 290
Topological Vector Spaces 293
Banach Spaces, Banach Algebras, Hilbert Spaces 294
Topology 297
General Topology 297
Algebraic Topology 301
Geometry 303
Geometries, Euclidean and Other 303
Convex Domains, Integral Geometry 305
Differential Geometry 305
Manifolds, Connections 312
Complex Manifolds 314
Algebraic Geometry 318
Numerical Analysis 322
Numerical Methods 322
Computing Machines 325
Probability 325
Statistics 329
Physical Applications 334
Mechanics of Particles and Systems 334
Statistical Thermodynamics and Mechanics . . 334
Elasticity, Plasticity 336
Structure of Matter 344
Fluid Mechanics, Acoustics 346
Optics, Electromagnetic Theory, Circuits . . . 354
Classical Thermodynamics, Heat Transfer . . 360
Quantum Mechanics 361
Relativity 367
Astronomy 368
Geophysics 370
Other Applications 371
Economics, Management Science 371
Programming, Resource Allocation, Games . . 373
Biology and Sociology 374
Information and Communication Theory . . . 374
Control Systems 375

AUTHOR INDEX

Abbasov, A. A. See Mirzadžanzade, A. H.
Adachi, R. ... 323
Ades, C. S. See Lee, L. H. N.
Adhikari, B. P.-Joshi, D. D. ... 329
Agranovič, Z. S.-Marčenko, V. A. ... 278
Ahlfors, L. See Beurling, A.
Akaike, H. ... 327
Ålander, M. ... 288
Albert, A. A. ... 246
Alder, K.-Winther, A. ... 324
Alekseev, S. A. ... 340
Alexandrow, A. D. ... 304
Altman, M. ... 242, 297
Amemiya, A. See Kotani, M.
Amsler, M. H. ... 373
Anderson, T. W. ... 332
Andrunakievič, V. A. ... 244
Arai, T. See Ishiguro, E.
Arens, R. ... 260
Arnous, E. ... 366
Arsove, M. G. ... 259
Arus, L. ... 359
Arutyunyan, N. H.-Čobanyan, K. S. ... 342
Aržanyh, I. S. ... 365
Aržanyh, I. S.-Bondarenko, B. A. ... 336
Aumann, G. ... 334
Auslander, M.-Buchsbaum, D. A. ... 249
Avalishvili, L. E. ... 348
Avališvili, L. E. ... 348
Bader, R. See Sörensen, W.
Baer, R. ... 248
Bak, M. A.-Petržak, K. A.-Romanov, Yu. F. ... 359
Banaschewski, B. ... 297
Banerji, S. K. ... 346
Barbălat, I. ... 275
Barner, M. ... 307
Barnes, E. S. ... 251
Barrett, W. ... 322
Basch, A. ... 334
Basov, V. P. ... 273
Batschelet, E.-Grün, F. ... 323
Batyrev, A. V. ... 259
Bauer, F. L. ... 322
Beckmann, M. J. ... 374
Beckmann, P.-Franz, W. ... 264
Bečvář, J. ... 254
Behrend, F. A. ... 298
Belardinelli, G. ... 279
Bellman, R. ... 274
Benado, M. ... 243, 248
Bereis, R.-Brauner, H. ... 305
Berencz, F. ... 365
Berezanskiĭ, Yu. M. ... 288
Berezin, F. A. ... 292
Berger, J.-Snell, J. L. ... 371
Bernardi, S. D. ... 258
Bers, L. ... 283
Beurling, A.-Ahlfors, L. ... 258
Bhattacharyya, M. N. See Krishna Iyer, P. V.
Bhattacharyya, P. K. ... 333
Bing, R. H. ... 300
Bininda, N. ... 244
Biot, M. A. ... 343, 359
Birkhoff, G. ... 296
Blanchard, A. ... 316
Blaschke, W. ... 324
Blehman, I. I. ... 276
Bliss, C. I. ... 330
Bocchieri, P.-Loinger, A. ... 362
Bodiou, G. ... 362
Bogoliubov, N. N. ... 362
*Bohm, D. ... 361
Bokštein, M. F. ... 301
Bomze, J. ... 358
Bondarenko, B. A. See Aržanyh, I. S.
Bonferroni, C. E. ... 254, 372
Bonfiglioli, L. ... 324
Bordoni, P. G. ... 336
Borgardt, A. ... 366
Borghi, D. C. ... 273
Borisov, Yu. F. ... 305
Borkowski, L. ... 240
Born, J. S. See Horvay, G.
Bott, R. ... 291
Bourgin, D. G. ... 301
Bradt, R. N.-Karlin, S. ... 332
Bradt, R. N.-Johnson, S. M.-Karlin, S. ... 333
Braun, W. H. See Mirels, H.
Braunbek, W. ... 276
Braunbek, W.-Sauter, E. ... 276
Brauner, H. See Bereis, R.
Brelot, M.-Choquet, G. ... 261
*Brillouet, G. ... 348
Brillouin, L. ... 362
Brodskiĭ, M. S. ... 274
Brooker, R. A.-Sumner, F. H. ... 322
Brookes, B. C. ... 374
Brown, E. H. ... 337
Brown, G. E.-De Dominicis, C. T. ... 364
Brunk, H. D.-Ewing, G. M.-Utz, W. R. ... 289
*Buchdahl, H. A. ... 354
Buchdahl, H. A. ... 355
Buchsbaum, D. A. See Auslander, M.
Bunimovich, V. I. ... 360
Busemann, H. ... 305
Bushko-Zhuk, M. M. ... 304
Cabannes, H. ... 351
Camia, F. ... 360
Campagne, C.-Driebergen, C. ... 373
Campanato, S. ... 283
Campbell, R. ... 267
Cap, F. ... 356
Carleson, L. ... 252
Carlitz, L. ... 253
Carrier, G. F.-DiPrima, R. C. ... 349
Cartier, P. ... 247
Case, K. M. ... 366
Castañeda, J. ... 374
Castiglioni, P. ... 372
Cazenave, R. ... 359
Černyaev, M. P. ... 306
Černý, G. G. ... 352, 353
Cesari, L. ... 256
Chakravorty, J. G. ... 338, 339, 341
Chandra Das, Sisir. ... 338, 339
Chandrasekharan, K. ... 268
Chang, C. C. ... 243
Chern, Shiing-shen. ... 314
Chester, W. ... 353
Chihara, T. S. ... 263
Chisini, O. ... 254
Choquet, G. See Brelot, M.
Chu, Boa-Teh. ... 343
Chuck, Wong. See MacDonald, W. M.
Church, A. ... 239
Ciarrapico, L. ... 298
Clatworthy, W. H. ... 332
Clunie, J. ... 259
Čobanyan, K. S. See Arutyunyan, N. H.
Cohen, A. C., Jr. ... 330
Constantinescu, P. ... 375
Cornfield, J.-Tukey, J. W. ... 331

(*Continued on cover 3*)

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FOUNDATIONS, THEORY OF SETS, LOGIC

★**Reidemeister, Kurt. Raum und Zahl.** Springer-Verlag, Berlin-Göttingen-Heidelberg, 1957. vii+151 pp. DM 19.80.

This brief volume, addressed to educated laymen, explores the nature of mathematical reasoning. It comprises ten chapters of uneven length and nonuniform style. These are essays or lectures rather loosely related by the common topic of elementary geometry and a common philosophical concern in intuitional perception (Anschauung) and reflective reasoning (Denken). The author's aim, like that of Pestalozzi, 1826, is a critical defense of these traditional commonsense features of mathematics, which modern formalism might be held to dethrone. Intuitional perception is "the apprehension of the individual items in relation to a general ordering"; reflective reasoning is "the process of combining and analyzing". Menon's dialogue, eliciting the incommensurability of the diagonal of a square in terms of a side, is given in toto. Linkages are treated. Kant's ontological argument for the reality of space from the existence of enantiamorphs is explored at considerable length. The foundations of the calculus, combinatory topology (of knots), number fields and geometry (in connection with construction of regular polygons), a chapter of appreciative eulogy of Gauss, all of these and other topics are discussed. A closing plea, "prolegomena to a critical philosophy", seeking to harmonize the views of positivists and existential formalists, urges that mathematics is all that these claim separately, but is more besides. *A. A. Bennett.*

Mostowski, Andrzej. Mathematical logic at the international congress of mathematicians in Amsterdam. Studia Logica **4** (1956), 245–253. (Polish)

A report. *J. Łoś* (Toruń).

Łuszczewska-Romahnowa, Seweryna. Induction and probability. Studia Logica **5** (1957), 71–90. (Polish. Russian and English summaries)

Some interesting remarks on induction, of purely philosophical character. *J. Łoś* (Toruń).

Czerwiński, Zbigniew. The problem of probabilistic justification of enumerative induction. Studia Logica **5** (1957), 91–107. (Polish. Russian and English summaries)

After making some critical remarks on the well-known Keynes method of justifying enumerative induction, the author tries to obtain a partial solution of this problem in two directions. First, he shows that if n choices from a finite set A have produced only elements with the property B, then the conclusion "every A is B" is more probable than "there are k/l B's in A", where $k<l$. Then he regards enumerative induction as a statistical test, and shows that it fulfills the strongest criteria required for such tests. *J. Łoś* (Toruń).

Takeuti, Gaisi. On the theory of ordinal numbers. J. Math. Soc. Japan **9** (1957), 93–113.

The author first introduces a logical system which is a certain type-theory formulated in a manner similar to Gentzen's, and for which Gentzen's Hauptsatz holds. In this system he sets up an axiomatic theory of ordinals and shows that within this resulting system it is possible to build a model of von Neumann-Fraenkel set-theory.

L. N. Gál (Ithaca, N.Y.).

Richardson, Moses. Relativization and extension of solutions of irreflexive relations. Pacific J. Math. **5** (1955), 551–584.

If H is a subgraph of the graph G, then the juncture $\bar{H}$ of H relative to G is obtained by adding to H all the arcs of G which join vertices of H. If $\bar{H}=H$, H is termed a conjunct subgraph of G. A solution V of G is termed an extension of a solution V_0 of H if $V \supseteq V_0$; V_0 is said to be relativized from V. The paper contains some sufficient conditions for relativizations and extensions; in this connection, twelve theorems and ten lemmas are proved. E.g., Th. 1: If G_0 is a conjunct subgraph of G and V is a solution of G, then $V \cap D_0$ is a solution of G_0, provided $D^{-1}(y, G) \subseteq D_0$ (here D_0 denotes the set of vertices of G_0; $D^{-1}(x, H)$ denotes the set of all vertices $y \in H$ satisfying $y > x$). Some of these theorems (e.g., 2 and 11) are mutually dual. *Đ. Kurepa* (Zagreb).

Kalmár, László. Solution of a problem of K. Schröter concerning the definition of the concept of the general recursive function. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **5** (1955), 103–127. (Hungarian)

Fitch, Frederic B. Recursive functions in basic logic. J. Symb. Logic **21** (1956), 337–346 (1957).

The author's earlier system K^* of basic logic [same J. **18** (1953), 317–325; MR **15**, 924] is here presented in an alternate fashion through replacement of the rule $[E]$: $ab \rightarrow Ea$ by the rule $[F]$: $abc \rightarrow Fa(bc)$. Negation is obtained as before by the author [ibid. **19** (1954), 29–36; MR **16**, 2]. General recursive functions are shown to be definable in K^* with functional notation retained. Let '$(_0ab)$' stand for 'b', '$(_1ab)$' stand for 'ab', '$(_2ab)$' for '$a(ab)$', etc., and say that 'p' is a "natural numeral" having index m if for all 'a', 'b', and 'c' there holds (in K^*): $pabc \leftrightarrow [c=(_mab)]$. A function λ (on a set of n-tuples of natural numbers to natural numbers) is said to be the index of the U-expression 'f' provided $fp_1 \cdots p_n abc \leftrightarrow [c=(_{\lambda(x_1 \ldots x_n)}ab)]$ for all numerals 'p_1', $\cdots$, 'p_n' having indices $x_1, \cdots, x_n$ for which $\lambda(x_1, \cdots, x_n)$ is defined. Then (among other results) every primitive recursive function is the index of some U-expression; every partial recursive function is the index of some U-expression; every general recursive relation is the index of some U-expression. *R. M. Baer.*

Grzegorczyk, A. On the definition of computable functionals. Fund. Math. **42** (1955), 232–239.

Computable functionals are here defined in two different ways: first by an inductive mathematical definition and secondly by a meta-mathematical definition. It is then shown that these two definitions are equivalent.

L. N. Gál (Ithaca, N.Y.).

Grzegorczyk, A. Computable functionals. Fund. Math. **42** (1955), 168–202.

The paper starts by giving an inductive definition of the notion of a computable functional in a manner analogous to the definition of a general recursive function. Following a section containing some basic properties of these functionals is the proof of a uniformity theorem. The last section contains applications of computable functionals to analysis. Two of the questions raised here are answered in the paper reviewed below, which also contains an alternative proof of the uniformity theorem.

L. N. Gál (Ithaca, N.Y.).

Kleene, S. C. A note on computable functionals. Nederl. Akad. Wetensch. Proc. Ser. A. **59**=Indag. Math. **18** (1956), 275–280.

This paper is devoted to several remarks concerning results in the paper reviewed above. Here, three kinds of computable functionals were considered: functionals computable in the sense of Banach and Mazur (op. cit., Property 8, p. 175), those of class $\mathscr{K}$ (op. cit., section 1), and general recursive functionals [Kleene, Introduction to metamathematics, Van Nostrand, New York, 1952; MR **14**, 525]. The author points out that computability in the first sense is weaker than in the other two senses, which are equivalent. It is shown that the uniformity theorem on p. 180 of Grzegorczyk's paper is a corollary of Kleene's normal form theorem (op. cit., p. 292) and Brouwer's fan theorem [Brouwer, Akad. Wetensch. Amsterdam, Proc. **27** (1924), 644–646; Math. Ann. **97** (1926), 60–75]. An example is given of a functional, not general recursive, but computable in the first sense and expressible in the form $\varphi(\alpha_1, \cdots, \alpha_l, x_1, \cdots, x_n) = \lim_{m\to\infty} \psi(\tilde{\alpha}_1(m), \cdots, \tilde{\alpha}_l(m), x_1, \cdots, x_n)$ with ψ general recursive. It is remarked that the scheme

$$\varphi(\alpha, 0, x) = \chi(\alpha, x)$$

$$\varphi(\alpha, n+1, x) = \psi(\lambda l \varphi(\alpha, n, l), x)$$

does not lead outside the class of general recursive functionals.

E. J. Cogan (Hanover, N.H.).

Meschkowski, Herbert. Zur rekursiven Funktionentheorie. Acta Math. **95** (1956), 9–23.

This paper is mainly concerned with (recursively) uniformly convergent sequences of functions which are defined for the rationals and are uniformly continuous over the rationals. The values at recursive real numbers (actually also at real numbers defined by free choice sequences) are obtained from (the recursive modulus of) continuity. As observed by Grzegorczyk (G) [see the paper reviewed second above, esp. footnote 12 on p. 201], this class of functions is coextensive with the class $\mathscr{K}$ of (G). The results of the present paper were anticipated in (G). Large parts of the paper are confused because the author does not realize that the classes of real numbers definable by (general) recursive, recursively convergent sequences of rationals and by recursive sections are (i) extensionally equivalent and (ii) not effectively equivalent, i.e., there does not exist a recursive functional which maps an element of the former class into a corresponding element of the latter; (ii) is obvious, and (i) was observed by R. Robinson in his review in J. Symb. Logic **16** (1951), 280–282. The author raises a question which also appears on p. 201 of (G): does there exist a function of this class which does not attain its maximum at a recursive real number? Such a function is got from Kleene's predicate $A(w)$ [Proc. Internat. Congress Math., Cambridge, Mass., 1950, v. 1, Amer. Math. Soc., Providence, R.I., 1952, pp. 679–685 (p. 683, line 3 from bottom) MR **13**, 422] as follows. Let $a_1 \cdots a_n$ be the immediately secured (i.s.) sequences of $A(w)$ consisting of 0 and 1; suppose $a_m = 1$ and $m < r \leqq n \to a_r = 0$. Then (i) the sets of intervals $[\cdot a_1 \cdots a_m, \cdot a_1 \cdots a_m + 2^{-n})$ (in binary notation) are recursive and non-overlapping, (ii) every recursive sequence of 0 and 1 has an i.s. initial segment, but not every non-recursive sequence, (iii) each interval (except the one with left-hand endpoint 0 and the one with right-hand endpoint 1) has immediate neighbours on both sides: if $a_1 \cdots a_{m-1}$ $01 \cdots 1$ $(=a)$ with a tail of k successive 1's is the i.s. initial segment of $a_1 \cdots a_{m-1}\, 0\dot{1}$ (1 periodic), the left neighbour of the interval of (i) is $[\cdot a, \cdot a_1 \cdots a_m)$, and similarly a right neighbour. Define

$$f(0) = f(1) = -1,\ f(\cdot a_1 \cdots a_m) = \max(-2^{-n}, -2^{-(m+k)}),$$

with linear interpolation inside intervals; f is (recursively) uniformly continuous, $\max f(x) = 0$ for $0 \leqq x \leqq 1$, but this is not attained inside any interval of (i), i.e., not for a recursive argument. Since, by (G) p. 196, isolated maxima are attained at recursive arguments, the complementary set with respect to [0, 1] of the intervals above is perfect, nowhere dense, and consists exclusively of non-recursive real numbers. The author's question is also answered in the last paragraph of a note by Lacombe [C.R. Acad. Sci. Paris **241** (1955), 1250–1252; MR **17**, 701].

G. Kreisel (Reading).

★ **Mostowski, Andrzej. Development and applications of the "projective" classification of sets of integers.** Proceedings of the International Congress of Mathematicians, 1954, Amsterdam, vol. III, pp. 280–288. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam, 1956. $7.00.

This paper is essentially a preview of four more detailed papers by the author [I, Fund. Math. **42** (1955), 271–275; II, ibid. **42** (1955), 259–270; III, ibid. **42** (1955), 125–140; MR **17**, 816, 815, 225; IV, ibid. **44** (1957), 12–36], and part of another paper [V, ibid. **44** (1957), 37–51]. — The "projective" classification considered is the arithmetic hierarchy developed independently by Kleene [Trans. Amer. Math. Soc. **53** (1943), 41–73; MR **4**, 126] and the author [Fund. Math. **34** (1947), 81–112; MR **9**, 129], which is variously regarded as an analogue to Lusin's projective hierarchy or the finite Borel hierarchy. The development consists of (i) the extension of the classification to functions as in I, (ii) a brief remark on the extension of the hierarchy to infinite ordinals, as developed by the author [Studia Philos. **4** (1951), 237–274] and Kleene [Trans. Amer. Math. Soc. **79** (1955), 312–240; MR **17**, 4]; while these extensions make sense only for constructive ordinals, the author now proposes an extension along the lines of ramified set theory [cf. Wang, J. Symb. Logic **19** (1954), 241–266; MR **16**, 661], which is possible for all ordinals of the second number class. The applications consist (i) in classifying, from the point of view of hierarchy theory, various concepts of analysis, as is done

in detail in II, (ii) in deductions from Kleene's [Nederl. Akad. Wetensch. Proc. **53** (1950), 800–802; MR **12**, 71] disjoint recursively enumerable, but not recursively separable sets, as in III and V. Finally, there is a section on generalized quantifiers, which is developed in IV: this section is of interest also independently of hierarchy theory. *G. Kreisel* (Reading).

Church, Alonzo. Binary recursive arithmetic. J. Math. Pures Appl. (9) **36** (1957), 39–55.

By "binary recursive arithmetic" the author means the theory of primitive recursive functions of natural numbers (i.e. nonnegative rational integers) having not more than two arguments. The author sets up a formalization of this sort of arithmetic, and makes some steps toward a proof that it is adequate, not only to define all the functions, but to express their properties within the formalism by means of free variables. The essential features of this formalization are as follows. There are four sorts of well formed formulas, viz., terms, unary (called by the author "singulary") functions, binary functions, and propositional forms. The atoms are an infinite set of variables of each sort, a constant term (0), three constant unary functions (successor, identity, and the function with 0 as a constant value), one constant binary function (viz., v such that $v(x, y)=y$). The operations consist of application of a unary function to a term to form a term; application of a binary function to two terms to form a term; an operation C, called composition, which combines a binary function f and two unary functions g_1 and g_2 to form a unary function h such that $h(x)=f(g_1(x), g_2(x))$; an operation R for combining a unary function f and two binary functions g_1, g_2 to form a binary function h such that $h(x, 0)=f(x)$, $h(x, y')=g_1(g_2(x, y), h(x, y))$; equality, which makes a propositional form out of two terms; and negation and implication as propositional connectives. The axioms agree with this interpretation, except that the replacement property of equality is assumed only in the form

$$x=y.\supset.g(x, z)=g(y, z).$$

The rules are modus ponens, a substitution rule for each of the four kinds of variables, and mathematical induction. The author shows that a variety of elementary functions are definable in this system, including the binary function $w(x, y)=x$, sum, predecessor, difference, triangular number, and functions associated with the ordered pair. He concludes that his C and R are sufficient to generate all unary and binary primitive recursive functions from his four initial functions. However the system is not without anomalies. Thus the author conjectures that

$$y=z.\supset.g(x, y)=g(x, z)$$

is unprovable, although any special case obtained by giving x a numerical value is provable; if so then there is no general way of expressing the converse of a binary function, or the application of a binary function to a term to produce a unary function. Thus the anomalies, if verified, are due to the somewhat artificial restrictions; even so they would furnish an interesting example of ω-incompleteness. *H. B. Curry* (University Park, Pa.).

Kubiński, Tadeusz. On a method of constructing modal logics. Studia Logica **4** (1956), 213–240. (Polish. Russian and English summaries)

For a very simple logic of first degree (without quantifiers and variables, with constants only) the author defines in its metalanguage the modal connectives. 'α is necessary with respect to β' means that β is consistent and α is deducible from β; 'α is impossible with respect to β' means that β is consistent and non-α is deducible from β; 'α impossible with respect to β' means that α is neither necessary not impossible with respect to β. It appears that the author considers such methods of defining modal connectives to be of great generality. There are some errors in this paper. *J. Łoś* (Toruń).

★ **Kanger, Stig. Provability in logic.** Stockholm Studies in Philosophy 1. Almqvist and Wiksell, Stockholm, 1957. 47 pp. Sw. Cr. 10.00.

The author presents a Gentzen-type approach to a form of predicate logic with an extension to some modal logics. He states that his aim is to give a technique of logical proof easier to master than that usually encountered in textbooks on logic. A Gentzen-type predicate calculus LC is described, based on a language L. Sequents in L are of the form $\Theta\rightarrow\Gamma$, where Θ and Γ may have infinitely many members. A valuation process is given for L and is used to provide definitions for the concepts of logical truth and validity. Each of these concepts is proved to correspond to provability in LC. A proof procedure is described by means of which a tree-form proof can effectively be obtained for any formula which is provable. A variation of the Löwenheim-Skolem theorem is proved. The valuation process is extended to cover certain modal logics. Special mention is made of the Lewis propositional calculi S4 and S5 and the Feys calculus t, for each of which a Gentzen-type calculus is explicitly given and a decision procedure obtained. At several places in the proof, nonconstructive methods are used and, in one place, a heuristic, set-theoretic principle is employed. Several proofs, especially in the discussion of modal systems, are given only in outline. Connections between the Gentzen calculi and other formalizations of the same calculi are often left vague or are stated without proof. In particular, no proof is given of the fact that a formula A is a theorem of S5 if and only if $\rightarrow A$ is provable in the corresponding Gentzen calculus. Details of such a proof would probably be of interest. In his introduction, the author seems to imply that, with a standard axiomatic formalization of predicate calculus (in contrast to a Gentzen-type formalization), there is no effective method for obtaining a proof of a formula which is known to be provable. It appears to the reviewer that there is such a method based on a consideration of an enumeration of all possible proofs in the calculus. The Gentzen approach would, however, lead in general to the much more rapid finding of a proof of the considered formula. If detailed proofs could be given, the results obtained and the methods used in the paper would seem to be of considerable interest. Author's correction: p. 22, l. 23, after "where" insert "either Γ' [Θ'] contains less than i members and". *R. Harrop*.

Rose, Alan. Sur les éléments universels de décision. C. R. Acad. Sci. Paris **244** (1957), 2343–2345.

A universal decision element is a logical function φ such that all functions of one or two arguments are definable in terms of φ and the logical constants 0, 1, the definiens containing only one occurrence of φ. It is shown that any φ having the form $\varphi(p, q, r, s)=[p, \psi(q, r), \sim s]$ is a universal decision element, where $[a, b, c]$ is conditioned disjunction, and ψ is one of $\equiv$, $\not\equiv$, $\supset$, or $\not\supset$.

V. E. Beneš (Murray Hill, N.J.).

Borkowski, Ludwik. Recent investigations on the calculus of propositions. Studia Logica **5** (1957), 27–40. (Polish)

McNaughton, Robert. On the measure of normal formulas. Pacific J. Math. **7** (1957), 969–982.

Aussagenlogische Formeln ϕ in der disjunktiven Normalform $B_{11}B_{12}\cdots B_{1j_1} \vee B_{21}B_{22}\cdots B_{2j_2} \vee \cdots \vee B_{k1}B_{k2}\cdots B_{kj_k}$ (wobei jedes $B_{n\lambda}$ eine der vorgegeben Variablen $A_1, \cdots, A_d$ oder eine Negation hiervon ist) können „der Einfachheit nach" geordnet werden: für $j_1 \leqq j_2 \leqq \cdots \leqq j_k$ ordne man lexikalisch nach den Systemen $\langle j_1, \cdots, j_k \rangle$. Die Anzahl der 2^d Belegungen von ϕ mit Wahrheitswerten, die ϕ erfüllen, heisst das Mass $m(\phi)$. Auf Grund einer Schaltungsaufgabe für Rechenmaschinen sucht Verfasser nach einem praktischen Verfahren um "möglichst einfache" Formeln mit vorgegebenen Mass zu bestimmen. Um das Durchprobieren aller Fälle abzukürzen, werden einige Abschätzungen des Masses mit Hilfe der Systeme $\langle j_1, \cdots, j_k \rangle$ bewiesen; z.B. gilt unter gewissen trivialen Bedingungen

$$m(\phi) \geqq 2^{d-j_1} + 2^{d-j_2-1} + \cdots + 2^{d-j_k-(k-1)}.$$

P. Lorenzen (Kiel).

Mostowski, Andrzej. Concerning a problem of H. Scholz. Z. Math. Logik Grundlagen Math. **2** (1956), 210–214.

The problem in question is that proposed in J. Symb. Logic **17** (1952), 160, viz., to characterize those sets M of natural numbers (i.e., non-negative rational integers) such that, for some formula H of the first order predicate calculus with equality, k is in M just when H is satisfiable in a domain of k individuals. The author proves that for each function $f(n)$ of a class K of functions, which is like the class of primitive functions except that at each step all functions are truncated above at n, there is a formula H which has a model in a set of $n+1$ individuals if and only if $f(n)=0$. From this he deduces positive solutions to Scholz's problem in a number of special cases. [Cf. Asser, Z. Math. Logik Grundlagen Math. **1** (1955), 252–263; MR **17**, 1038.] *H. B. Curry* (University Park, Pa.).

Zubieta R., Gonzalo. Formal definitions of enumerability. Bol. Soc. Mat. Mexicana (2) **1** (1956), 49–56. (Spanish)

The author's purpose is to exhibit a certain relativity of the notion of enumerability. As an illustration of this point he argues that in Quine [Amer. Math. Monthly **44** (1937), 70–80] there are two non-equivalent possible definitions of ordered pair (x, y), viz. $\{\{x\}, \{x, y\}\}$ and $\{\{x\}, \{x, \{y\}\}\}$, and consequently different definitions of single-valued function and enumerability. The early part of the paper contains a notably clear exposition of certain standard theorems of predicate calculus, e.g., the Skolem-Löwenheim theorem after Henkin. *H. B. Curry.*

Rasiowa, H., and Sikorski, R. An application of lattices to logic. Fund. Math. **42** (1955), 83–100.

Let $\mathscr{S}$ be a sentential calculus, $\bar{\mathscr{S}}$ the corresponding sentential calculus with quantifiers over sentential variables, $\bar{S}$ the class of sentences of $\bar{\mathscr{S}}$. $\mathscr{S}$ determines an abstract algebra called an $\mathscr{S}$-algebra with algebraic operations corresponding to $+$, $\cdot$, $\rightarrow$ and the sentential operators of $\mathscr{S}$. These $\mathscr{S}$-algebras are relatively-pseudo-complemented lattices with a unit element e. Let A be an $\mathscr{S}$-algebra. Every formula $\alpha \in \bar{S}$ can be interpreted as an algebraic function $(A)\phi_\alpha$ in a natural way. (In this interpretation the quantifiers $\sum_{a_i}$ and $\prod_{a_i}$ are treated as infinite sums and products whenever they exist.) Let $\{x_i\}$ be an infinite sequence of elements of A, $\alpha \in \bar{S}$, a_i the sentential variables of S. If for the substitution of x_i for a_i in α all the sums and products corresponding to the logical quantifiers of α exist for $(A)\phi_\alpha$, then the sequence $\{x_i\}$ is called a valuation of α. $\{x_i\}$ is called a valuation of $\bar{S}$ if it is a valuation of every $\alpha \in \bar{S}$. A is called an $\bar{S}$-algebra if every sequence $\{x_i\}$ with $x_i \in A$ is a valuation of $\bar{S}$. It is then shown that for every $\alpha \in \bar{S}$, $\vdash\alpha$ if and only if $(A)\phi_\alpha = e$ for every $\bar{\mathscr{S}}$-algebra A. For the special calculi $\bar{\mathscr{P}}_\kappa$ (classical), $\bar{\mathscr{P}}_\lambda$ (Lewis), $\bar{\mathscr{P}}_\chi$ (Heyting), $\bar{\mathscr{P}}_\mu$ (modal), this equivalence can also be expressed in a topological form. It is also shown that in $\bar{\mathscr{P}}_\lambda$, $\bar{\mathscr{P}}_\chi$, $\bar{\mathscr{P}}_\mu$ (but not in $\bar{\mathscr{P}}_\kappa$) there exist infinite sequences of closed non-equivalent formulas. Lastly there is a theorem showing a connection between $\bar{\mathscr{P}}_\chi$ and $\bar{\mathscr{P}}_\lambda$. *L. N. Gál* (Ithaca, N.Y.).

Markov, A. A. Mathematical logic and numerical analysis. Vestnik Akad. Nauk SSSR **27** (1957), no. 8, 21–25. (Russian)

A discussion, in popular language, of the similarity of the algorithms of mathematical logic to those used in elementary calculations. Two concluding paragraphs point out that modern computing machines have already posed new problems in logic.

See also: Quantum Mechanics: Bohm. **Control Systems:** Moisil (second from last article).

ALGEBRA

Mal'cev, A. I. Subdirect products of models. Dokl. Akad. Nauk SSSR (N.S.) **109** (1956), 264–266. (Russian)

Necessary and sufficient conditions are obtained for the inclusion of a class of abstract models in the class of all its subdirect products. *V. E. Beneš.*

Dwinger, Ph. Some theorems on universal algebras. I, II. Nederl. Akad. Wetensch. Proc. Ser. A. **60**= Indag. Math. **19** (1957), 182–189, 190–195.

Some elementary results, relating the structure of universal algebras with permuting congruence relations to the structure of the lattice of congruence relations, are proved.

R. P. Dilworth (Pasadena, Calif.).

Combinatorial Analysis

Kotzig, Anton. On a combinatorial problem. Mat.-Fyz. Sborník Slovensk. Akad. Vied Umení **2** (1952), no. 3–4, 3–12, 29. (Slovak. Russian summary)

The author considers sequences $A=\{a_1, \cdots, a_n\}$ with the following properties: (α) the a_i are positive integers, $a_i \neq a_{i+1}$, $a_n \neq a_1$; (β) the a_i assume at most three different values. Given such a sequence A, let M be the set of positive integers $0 \leqq i \leqq n$, such that $a_{i-1}=a_{i+1}$ (set $a_0=a_n$, $a_{n+1}=a_1$). The author characterizes the possible sets M by the condition that if $M=\{x_1<x_2<\cdots<x_m\}$ then m is even and $n+\sum_{i=1}^{u} x_{2i} - \sum_{i=1}^{u} x_{2i-1} = 0 \pmod 3$, where $u=\frac{1}{2}m$. *R. Bolt* (Ann Arbor, Mich.).

Hawkins, David; and Mientka, Walter E. On sequences which contain no repetitions. Math. Student **24** (1956), 185–187 (1957).

A sequence S: $x_0, x_1, \cdots$, where $x_i=a$, b, or c, is constructed which has the property that it contains no adjacent identical blocks. The construction is as follows: Let $A=bacbcacbabcbaca$, $B=bacbabcbacbcacbaca$, $C=bacbcabacabcbaca$. Given a block U of the letters a, b, c, let $T(U)$ be a block obtained by replacing a by A, b by B and c by C. Then the blocks $B, T(B), T^2(B)$ form initial blocks of the desired sequence S. By choosing a suitable set of translates of S it is then possible to define an unending sequence with the same property of containing no adjacent identical blocks. A different unending sequence with this property was constructed by Morse and Hedlund [Duke Math. J. **11** (1944), 1–7; MR **5**, 202].

G. A. Hedlund (New Haven, Conn.).

Oakley, C. O.; and Wisner, R. J. Flexagons. Amer. Math. Monthly **64** (1957), 143–154.

An additional occurrence of $\binom{2n-1}{n}/(2n-1)$ [cf. Motzkin, Bull. Amer. Math. Soc. **54** (1948), 352–360; MR **9**, 489] is the number of certain permutations of n elements. For $n=3m$, a subclass containing $6\binom{4m-1}{m-1}/(4m-1)$ permutations leads to the enumeration of all regular n-order flexagons. The latter, discovered in 1939 by R. Feynman, A. H. Stone, B. Tuckerman and J. W. Tukey, are Moebius bands that can be folded and refolded into hexagonal shape by a definite procedure.

T. S. Motzkin (Los Angeles, Calif.).

Perfect, Hazel. Forms and functions. Math. Gaz. **41** (1957), 91–94.

The author comments on the differences between forms and functions after defining both terms precisely. As an example of these differences she uses formal power series to prove the so-called Wronski relations with a generality not possible from the analogous functional proof.

K. Goldberg (Washington, D.C.).

See also: Statistics: Clatworthy.

Linear Algebra

★**Pi Calleja, Pedro. Introduccion al algebra vectorial. [Introduction to vector algebra.]** With a preface by J. Rey Pastor. Universidad Nacional de Cuyo, Facultad de Ciencias, Buenos Aires, 1945. 137 pp.

An abstract treatment intended especially for engineers. The sixth and last chapter introduces tensors, linear operators, matrices, groups of transformations.

★**Thrall, Robert M.; and Tornheim, Leonard. Vector spaces and matrices.** John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London. xii+318 pp. $6.75.

As the dust cover states, this is a thorough systematic approach to the theory of vector spaces and matrices, which stresses up-to-date methods of mathematical reasoning. Beginning with the concept of a vector space over an arbitrary field, the basic properties of vector spaces are examined in the opening chapter. The second chapter is concerned with mappings both as linear transformations and as matrices, while the third studies systems of linear equations in the light of rank and of elementary transformations. Determinants are defined inductively by expansion of the first row, and their basic properties, as well as Cramer's rule, are obtained in Chapter 4. The next three chapters introduce the concept of a canonical form and the invariants which pertain to it. This leads to a discussion of equivalence and similarity for linear transformations, canonical forms for symmetric and skew-symmetric matrices, quadratic and Hermitian forms, orthogonal and unitary equivalence. Ch. 9 is concerned with the equivalence of matrices over the ring of integers and over the polynomial ring of a field. The necessary preliminaries on rings are found in the preceding chapter. Ch. 10 deals with the minimum function, the Cayley-Hamilton theorem and also the rational and irrational canonical forms for similar matrices. The last chapter deals with linear inequalities, convex sets, linear programming and matrix games.

In the preface the authors stress the fact that the book proceeds simultaneously at two levels, the concrete via matrices and the axiomatic via vector spaces. Though there is undoubted value in this dual approach the reviewer would question the adjectives used to describe them. In what sense is one more concrete than the other and why should matrices be regarded as unaxiomatic?

Throughout the book the arguments are given in fine detail and the volume is an excellent introduction to the theory of vector spaces, but it is to be doubted whether it would really be suitable for a reader unfamiliar with the summation sign Σ, which is defined on p. 37 after it has been proved that the dimension of the range of a linear transformation plus the dimension of the kernel is equal to the dimension of the domain.

One or two details might have been better expressed. On p. 70, T denotes a transformation, and a few lines further the same letter in the same font denotes the transpose of a matrix. On p. 96 it is proved that elementary matrices are non-singular, and immediately afterwards improper elementary matrices are defined albeit they are singular. In other words, improper elementary matrices are not elementary. This is at variance with the usage on p. 84, where improper elementary transformations are indeed elementary. The terms 'Hermite matrix' (p. 91) and 'Hermitian matrix' (p. 172) for distinct entities are likely to be confused by some readers. On p. 66 a null matrix is defined as one having no elements and the reader is left to consider the subtle distinction between a null row and a null column matrix.

In the reviewer's opinion this is an admirable book for a reader who already possesses some acquaintance with rings, fields, determinants and matrices. He will find it stimulating and lucid. Though these concepts are each defined and the book is self-contained, the reader without these prerequisites is likely to find the going rather hard.

D. E. Rutherford (St. Andrews).

Deprit, André. Quelques classes d'homomorphismes dans des espaces vectoriels. Ann. Soc. Sci. Bruxelles. Sér. I. **71** (1957), 5–43.

Let A and B be two vector spaces over a field A. A systematic study is made of the classes $\alpha(A, B)$ of homomorphisms of A into B whose kernel is finite-dimensional, $\beta(A, B)$ consisting of those homomorphisms whose range has finite co-dimension, and $\sigma(A, B)=\alpha(A, B)\cap\beta(A, B)$. In the case $A=B$, there are considered the classes $\nu(A)$ of those endomorphisms f such that the number of distinct subspaces of the form kernel (f^k) is finite; $\lambda(A)$, the class

made up of those f for which the number of distinct subspaces of the form $f^k(A)$ is finite; and $\rho(A)=\nu(A)\cap\lambda(A)$. Interconnections and characterizations are given. [For previous related work see M. Hukuhara, J. Fac. Sci. Univ. Tokyo. Sect. I. 7 (1954), 129–192; MR 16, 992.] These notions have previously been studied for continuous homomorphisms, where A and B are Banach spaces or locally convex topological spaces. The author's investigations are purely algebraic. *B. Yood.*

Altman, M. A generalisation of Jacobi's method for bilinear forms. Bull. Acad. Polon. Sci. Cl. III. 5 (1957), 99–104, X. (Russian summary)

This paper is essentially a reduction of a Hermitian matrix to its canonical form based upon a slight generalisation of Jacobi's method for reducing a matrix by equivalence. The language of the paper, however, is that of bilinear forms, vector spaces and linear functionals. *D. E. Rutherford* (St. Andrews).

Farahat, H. K. A note on the classical canonical form. J. London Math. Soc. 32 (1957), 178–180.

The author gives a constructive proof of a theorem proved in another form by P. R. Halmos [Finite dimensional vector spaces, Princeton, 1942, p. 162; MR 4, 11], namely: Let A be an n by n nilpotent matrix of index $q\leq n$. Then A is similar to the direct sum of matrices J_q and B, where B is some square matrix with $n-q$ rows and J_q is the q by q matrix all of whose elements are zero except for 1's immediately below the principal diagonal. *B. W. Jones* (Boulder, Colo.).

Motzkin, T. S.; and Taussky, Olga. Pairs of matrices with property L. II. Trans. Amer. Math. Soc. 80 (1955), 387–401.

This paper, equally interesting as to results and method, continues the study of pairs of matrices A, B (over an algebraically closed field F of characteristic $p\geq 0$) with the property L (that is, if α_i, β_i are the roots of A, B taken in a certain order, then $\lambda\alpha_i+\mu\beta_i$ $(i=1, \cdots, n)$ are those of $\lambda A+\mu B$ for all $\lambda, \mu \in F$), which the authors began in Part I [same Trans. 73 (1952), 108–114; MR 14, 236]. The roots of $\lambda A+\mu B$ are the roots in ν of the polynomial

$$f(\lambda, \mu, \nu)=|\nu I-\lambda A-\mu B|=\sum_{i=1}^{n} f_i\nu^i$$

whose ν-discriminant Δ (like the coefficient f_i) is a form in λ, μ; its roots in the ratio λ/μ are the so-called discriminant roots of the pencil $\lambda A+\mu B$. In the case of property L one has $f(\lambda, \mu, \nu)=\prod_{i=1}^{n}(\nu-\lambda\alpha_i-\mu\beta_i)$ and therefore $\Delta=\prod_{i<k}(\lambda\alpha_i+\mu\beta_i-\lambda\alpha_k-\mu\beta_k)^2$. Thus all discriminant roots are of even order (except when $\Delta\equiv 0$) and their number is $\leq\frac{1}{2}n(n-1)$ in the case L. Conversely, a pair A, B is said to have the property D if all its discriminant roots are of even order or if $\Delta\equiv 0$. Thus from L follows D. The first part of the paper establishes a sufficient condition for D to imply L. In the case of characteristic $p\neq 2$ that is so if no $\lambda A+\mu B$ has a root of multiplicity ≥ 3 or two different double roots. The proof is based on a discussion of the so-called characteristic curve C of the pencil $\lambda A+\mu B$ which, in the projective λ, μ, ν-plane over F, is represented by the equation $f(\lambda, \mu, \nu)=0$. In the case L it splits into straight lines. By counterexamples it is shown that the conditions are necessary if $p\neq 2$ and $p\neq 2, 3, 5$, respectively. If $p=2$ all pairs have property D, but not necessarily L. If $p\neq 2$ and $n=2$, then D (with $\Delta\not\equiv 0$) implies L and thus, apart from scalar multiples, there is only one matrix with double characteristic root in an L-pencil. In the subsequent part of the paper the method of the characteristic curve C is further employed to prove the following results. If in the pencil $\lambda A+\mu B$ all matrices are diagonable and $n\leq p$ (if $p\neq 0$), then A, B have the property L and they can be diagonalized simultaneously and therefore commute. (If $n>p$ the property L must be assumed for this result to hold.) If a single matrix in the pencil is not diagonable, then the theorem fails to be true. The paper concludes with three theorems in a way inverting the preceding result. Assuming the complex pair A, B commutative, it is shown that it is the limit of a sequence of simultaneously diagonable commuting matrix pairs (thus extending the known fact that every complex matrix is the limit of a sequence of diagonable matrices). Commutative A, B over F are in general diagonable; "in general" means that within the manifold of commutative pairs there is a lower-dimensional submanifold of non-diagonable pairs. [For a brief comment on the results of the paper see O. Taussky, Amer. Math. Monthly 64 (1957), 229–235, §§ 5–7, MR 19, 6.] *H. Schwerdtfeger* (Montreal).

Khan, N. A. Characteristic roots of semi-magic square matrices. Amer. Math. Monthly 64 (1957), 261–263.

A matrix $A=(a_{rs})$ is semi-magic if $\sum_r a_{rs}=\sum_s a_{rs}=S(A)$. Let the elements of A be positive real numbers, and let $f(x)$ be a rational function over the reals whose denominator is not singular for $x=A$, and such that $f(A)$ has positive elements. {The author neglects this last condition.} If λ is any characteristic root of A, then

$$|f(\lambda)|\leq f(S(A)).$$

R. J. Walker (Ithaca, N.Y.).

Guttman, Louis. Some inequalities between latent roots and minimax (maximin) elements of real matrices. Pacific J. Math. 7 (1957), 897–902.

Let A be an $m\times n$ real matrix, let $|\lambda|^2$ be the greatest characteristic root of AA', and let v be the value of the game whose matrix is A. The author uses the fundamental theorem of game theory to obtain $-|\lambda|n^{\frac{1}{2}}\leq v(mn)^{\frac{1}{2}}\leq|\lambda|m^{\frac{1}{2}}$. This result is used to obtain "quick information" about $|\lambda|$, and also about v, in certain cases. For example, $|\lambda|\geq pn^{\frac{1}{2}}$, where $p=\max_i \min_j a_{ij}$. {Reviewer's comment: All of the inequalities of the paper which do not involve v are consequences of the fact that $|\lambda|^2$ is at least as great as each diagonal element of AA' or of $A'A$.} *M. F. Smiley* (Iowa City, Ia.).

Parodi, Maurice. La localisation des zéros de la dérivé du polynome caractéristique d'une matrice. C.R. Acad. Sci Paris 244 (1957), 2764–2765.

As is well known, the zeros of the characteristic polynomial $f(z)$ of an nth order square matrix $A=(a_{ij})$ lie in the domain D comprised of the n circular disks C_i $(i=1, 2, \cdots, n)$ defined by $|a_{ii}-z|\leq P_i=\sum|a_{ij}|$, the summation being for $j=1, 2, \cdots, i-1, i+1, \cdots, n$. To this theorem the author applies Lucas' theorem and Jensen's theorem (mispelled Jansen thruout this paper) [see Marden, The geometry of the zeros of a polynomial in a complex variable, Math. Surveys, no. 3, New York, 1949, pp. 14–17; MR 11, 101] to obtain at once the following results: (1) The zeros of $f'(z)$, the derivative of $f(z)$, lie in D', the convex hull of D. (2) If the a_{ij} are all real, the complex zeros of $f'(z)$ lie within the n elliptical disks

$$(x-a_{ii})^2+2y^2\leq 2P_i^2 \quad (i=1, 2, \cdots, n).$$

M. Marden (Milwaukee, Wis.).

Greville, T. N. E. The pseudoinverse of a rectangular or singular matrix and its application to the solution of systems of linear equations. SIAM Newsletter **5** (1957), no. 2, 3–6.

Following E. H. Moore, the author defines the pseudo-inverse of a matrix A as follows. Let B be a matrix composed of a maximal set of linearly independent columns of A, C a matrix composed of a maximal set of linearly independent rows of A, and H^{-1} the square matrix common to B and C. Define the pseudo-inverse of B as $B^{-1}=(B^TB)^{-1}B^T$, of C as $C^{-1}=C^T(CC^T)^{-1}$ and that of A to be $A^{-1}=C^{-1}H^{-1}B^{-1}$. This reduces to the inverse of A if A is non-singular. The author shows that this definition yields an approximate solution of a system of equations for which no exact solution exists, extending results of Bjerhammar. *B. W. Jones* (Boulder, Colo.).

See also: Differential Geometry: Whitney. **Numerical Methods:** Bauer. **Mechanics of Particles and Systems:** Basch. **Programming, Resource Allocation, Games:** Fan, Glicks- berg and Hoffman.

Polynomials

See: Fields, Rings: Bininda; Niven and Warren. **Algebras:** Zykov. **Special Functions:** Fekete and Walsh. **Numerical Methods:** Barrett.

Partial Order, Lattices

Tamura, Takayuki. On a special semilattice with a minimal condition. J. Gakugei Tokushima Univ. Nat. Sci. Math. **7** (1956), 9–17.

A semilattice is a partially ordered set S in which any two elements have a l.u.b. The semilattice S is said to be dispersed if $a<c$ in S implies that $\{x: a\leqq x\leqq c\}$ is a finite chain. The distance from a to b, $b\leqq a$, is defined to be the length of the maximal chain connecting a and b. The main result of this paper shows that the unbounded, dispersed semilattices, in which each element contains a minimal element, are characterized by the set of the minimal elements and the distances from the elements of S to the minimal elements. { The reviewer doubts the validity of Lemma 4 without some additional assumption implying the existence of upper bounds used in the proof.}

J. Hartmanis (Columbus, Ohio).

Martić, Ljubobir. Sur les généralisations du treillis libre. C. R. Acad. Sci. Paris **244** (1957), 1593–1595.

The number of elements in the free semi-lattice [latticoid, Kimura, J. Sci. Gakugei Fac. Tokushima Univ. **1** (1950), 11–16; MR **13**, 425] generated by n elements is investigated. It is shown that for $n=1$, 2, 3, and 4 the orders of the free semilattice are 3, 12, 51, and 310.

R. P. Dilworth (Pasadena, Calif.).

Wolk, E. S. Dedekind completeness and a fixed-point theorem. Canad. J. Math. **9** (1957), 400–405.

As a generalization of the notion of lattices, the author introduces the concept of a uniform partially ordered set and proves analogues of theorems of MacNeille [Trans. Amer. Math. Soc. **42** (1937), 416–460], Tarski [Pacific J. Math. **5** (1955), 285–309; MR **17**, 574], and Davis [ibid. **5** (1955), 311–319; MR **17**, 574]. *R. P. Dilworth.*

Dwinger, Ph. On the axiom of Baer in distributive complete lattices. Nederl. Akad. Wetensch. Proc. Ser.A. **60**=Indag. Math. **19** (1957), 220–226.

The author shows that the condition (B_1), $y<z\leqq\bigcup A$ implies $y\cap\bigcup A'<z\cap\bigcup A'$ for some finite subset A' of A, is equivalent to infinite join distributivity in complete distributive lattices. A generalization of (B_1) which is equivalent to complete distributivity is also given.

R. P. Dilworth (Pasadena, Calif.).

Benado, Mihail. Rectification à mon travail "Les ensembles partiellement ordonnés et le théorème de raffinement de Schreier. II." (Théorie des multistructures). Czechoslovak Math. J. **6(81)** (1956), 287–288.

An error in an earlier paper by the same author [same J. **5(80)** (1955), 308–344; MR **17**, 937] is corrected. Hypothesis $\mathscr{M}$III (weakened absorption law) must be strengthened, but by a consequence of $\mathscr{M}1$ and $\mathscr{M}2$ of the same paper. *G. Birkhoff* (Cambridge, Mass.).

Kolibiar, Milan. Characterization of a lattice in terms of a ternary operation. Mat.-Fyz. Časopis. Slovensk. Akad. Vied **6** (1956), 10–14. (Czech. Russian summary)

Let M be a set containing at least the two elements O, I. It is shown that if T is a subset of $M\times M\times M$ satisfying specified postulates, and if $[a, b, c]\to(a, b, c)\in M$ is a function from T to M, also satisfying specified postulates, then setting $a\cap b=(a, O, b)$, $a\cup b=(a, I, b)$ imposes on M a lattice structure, with greatest and least elements I and O, in which

$$(*)\quad (a\cap b)\cup(b\cap c)\cup(c\cap a)\leqq(a, b, c)\leqq(a\cup b)\cap(b\cup c)\cap(c\cup a).$$

Any lattice with an O and I may be so obtained, by taking as T the set of triples $[a, b, c]$ for which the first and third members of (*) are equal, and setting $(a, b, c)=$ this common value. An example shows, however, that for suitable choice of the ternary operation strict inequality may hold in (*) (for a non-distributive lattice), and that the same lattice may be defined by different ternary operations. *From the author's summary.*

Chang, C. C. On the representation of α-complete Boolean algebras. Trans. Amer. Math. Soc. **85** (1957), 208–218.

The author studies conditions for an α-complete Boolean algebra to be representable as an α-homomorphic image of an α-field of sets. If $\alpha=\aleph_0$, such a representation is always possible [L. Loomis, Bull. Amer. Math. Soc. **53** (1947), 757–760; MR **9**, 20], but if $\alpha>\aleph_0$, there exist α-complete algebras which are not α-representable [R. Sikorski, Fund. Math. **35** (1948), 247–258; MR **10**, 437]. If α is any infinite cardinal number and A is a Boolean algebra, let $R_\alpha(A)$ be the set of elements x in A for which there exists a doubly indexed family $\{a_{ij}|i\in J, j\in I\}\subseteq A$ of cardinality at most α, which satisfies (1) $\prod_{j\in J}a_{ij}=0$ for all $i\in I$ and (2), for each $f\in J^I$, the set $\{a_{if(i)}|i\in I\}$ either contains x or a complementary pair of elements of A. It is shown that if A is α-complete, then $R_\alpha(A)$ is an α-complete ideal which acts as a radical for the property of α-representability: for any α-ideal N of A, the factor algebra A/N is α-representable if and only if $N\supseteq R_\alpha(A)$. Since $R_{\aleph_0}(A)=\{0\}$ always, this result includes Loomis' theorem. Finally, the author gives several alternative characterizations of $R_\alpha(A)$. In methodology, the paper is close to the work of Loomis cited above. *R. S. Pierce.*

See also: Foundations, Theory of Sets, Logic: Richardson; Rasiowa and Sikorski. **Algebra:** Dwinger. **Groups and Generalizations:** Benado. **Optics, Electromagnetic Theory, Circuits:** Povarov.

Fields, Rings

Lamprecht, Erich. Bewertungssysteme und Zetafunktionen algebraischer Funktionenkörper. III. Math. Ann. **132** (1957), 373–403.

For any extension K/k of finite type of a field k, denote by $P(K)$ the set of prime divisors of K/k (classes of equivalent discrete valuations of K/k); a prime divisor M in $P(K)$ is said to be trivial on a subfield K' of K if the valuations of the class M are trivial on K'. Let A/k be a field of algebraic functions of 2 variables over k; let K, K' be two distinct subfields of A, of transcendance degree 1 over k, algebraically closed in A; A is then algebraic over KK'. The author introduces the set $P(K, K')$ of prime divisors of A/k which are trivial on K' but not on K. For each V in $P(K, K')$, let $R(V)$ be the residue field of V, which is of transcendance degree 1 over k; each prime divisor v in $P(R(V))$ determines (by the process of composition of valuations) a class of equivalent discrete valuations of rank 2 of A/k, or, what amounts to the same, a place of A/k; let $b(K, K')$ be the set of places obtained in this manner. The author introduces a notion of divisor relative to $b(K, K')$; to every element $\neq 0$ of A there is associated such a divisor. There is a simultaneous approximation theorem for any finite set of places in $b(K, K')$. On the other hand, there is a natural mapping $f_{K,K'}: b(K, K') \to P(K) \times P(K')$. If (w,w') is in $P(K) \times P(K')$, the pair (w, w') is called permutable if there is a bijection of $f_{K,K'}^{-1}(w, w')$ on $f_{K',K}^{-1}(w', w)$ such that corresponding places have isomorphic residue fields. It is proved (as a special case of a more general result) that, if k is perfect, then there is an integer N such that, for almost all w in $P(K)$, all but at most N of the elements of $P(K')$ are permutable with w.

In case k is a finite field, the author has introduced the zeta function

$$Z(s; K, K') = \prod_{V \in B(K,K')} \zeta(R(V); s),$$

where $\zeta(R(V); s)$ is the zeta function of the field $R(V)$ [Math. Ann. **131** (1956), 313–335; MR **18**, 107]. In the present paper, he legitimatizes this definition by showing that the product converges in the domain $\Re(s)>2$; it is proved that the Z-function can be continued analytically at least in the region $\Re(s)>3/2$; assuming that it can be continued in the region $\Re(s)>1$, the number, locations and orders of the zeros and poles it may have in this region are invariants of A (they do not depend on K and K'); in some examples which were computed by the author[Math. Z. **64** (1956), 47–71; Arch. Math. **6** (1955), 266–274; MR **17**, 947, 350], one actually obtains in this manner invariants of the field A. *C. Chevalley.*

★ **Bininda, Norbert. Die Lösung der Gleichungen fünften Grades mit Hilfe der Ikosaedergruppe.** Dissertation, Ludwig-Maximilians-Universität zu München, 1955. vii+47 pp. (mimeographed)

Felix Klein [Vorlesungen über das Ikosaeder und die Auflösung der Gleichungen von fünften Grade, Teubner, Leipzig, 1884] showed that for every extension of an algebraic number field whose Galois group is the alternating group on five letters a generating equation can be so chosen that the Galois group can be represented by a certain group of linear transformations. The author proves this in a modern way for arbitrary ground fields of characteristic >5, and gets similar results for fields of characteristics 2, 3, or 5. *G. Whaples.*

Rim, Dock Sang. Relatively complete fields. Duke Math. J. **24** (1957), 197–200.

The author proves the following. (1) A field k is relatively complete (i.e. Hensel's lemma holds) with respect to a non-archimedean valuation if and only if this valuation has only one extension to the algebraic closure of k. (2) If k is not algebraically closed, and is relatively complete with respect to a rank one valuation, then every subfield for finite rank under k is relatively complete for the induced valuation. (This had been proved by Kaplansky and Schilling [Bull. Amer. Math. Soc. **48** (1942), 744–747; MR **4**, 71] under special assumptions on the field k.) (3) Let the algebraic closure of k be of finite degree over k. Then k is relatively complete for a valuation if and only if its residue class field is real-closed.

G. Whaples (Bloomington, Ind.).

Andrunakievič, V. A. Biregular rings. Mat. Sb. N.S. **39(81)** (1956), 447–464. (Russian)

A ring is called H-semi-simple when all its homomorphic images are semi-simple. The following generalisation of a theorem of Waddell [Duke Math. J. **19** (1952), 623–627; MR **14**, 348] is shown: An H-semi-simple ring is a discrete sum of simple rings with identities if and only if the annihilators of its maximal (two-sided) ideals are $\neq 0$. An ideal B of an arbitrary ring A is biregular when every principal ideal (b) of A with $b \in B$, considered as a ring, has an identity. Then [see also Brown and McCoy, Proc. Amer. Math. Soc. **1** (1950), 165–171; MR **11**, 638] every ring A contains one and only one maximal biregular ideal $B(A)$ and $B(A_n)=[B(A)]_n$, where A_n denotes the full ring of matrices of rank n over A. Moreover, B^* being the annihilator of $B(A)$, it is shown that $A=B(A)\oplus B^*$ if and only if the maximal biregular ideal of every principal ideal of A is a principal ideal itself. From the latter theorem the author derives decompositions into direct sums of a ring A which satisfies certain minimal or maximal conditions. *F. J. Terpstra* (Pretoria).

Andrunakievič, V. A. Antisimple and strongly idempotent rings. Izv. Akad. Nauk SSSR. Ser. Mat. **21** (1957), 125–144. (Russian)

The author calls the unique minimal nonzero ideal C of a subdirectly irreducible ring R the heart of R; C is either nilpotent or idempotent. If all subdirectly irreducible homomorphic images of a ring A have nilpotent hearts, then A is called antisimple, and it is shown that A is antisimple if and only if no ideal of A has a nonzero homomorphic image which is a simple ring. The antisimple radical of a ring R is introduced as the (unique) maximal antisimple ideal; it is the intersection of all semi-maximal prime ideals P of R, i.e. prime ideals P covered by some ideal of R. The antisimple radical is a radical in the sense of Kuroš [Mat. Sb. N.S. **33(75)** (1953), 13–26; MR **15**, 194]. A ring B is said to be strongly idempotent if all ideals of B are idempotent; this is equivalent to the fact that all homomorphic images of B are subdirect sums of subdirectly irreducible rings with idempotents hearts and also to the fact that its ideals are intersections of semi-maximal prime ideals. *L. Fuchs* (Budapest).

Leptin, Horst. Linear kompakte Moduln und Ringe. II. Math. Z. **66** (1957), 289–327.

The author continues his study [Math. Z. **62** (1955), 241–267; MR **16**, 1085] of the structure of linearly compact rings, especially rings "linearly compact in the narrow sense": inverse limits of discrete rings with minimum conditions. If R is one of the latter class of rings, we may quote the following sample theorems. If R is commutative, its radical N satisfies $\bigcap_{i=1}^{\infty} N^i=0$. If instead R has a unit, the upper and lower (transfinite) Loewy series of R have the same length. By generalizing the known power series examples, the author shows that this length can be arbitrary. R is a complete direct sum of indecomposable left ideals generated by idempotents. If, besides, R is primary (that is, modulo its radical it has no proper closed ideals) then R is a matrix ring over a completely primary ring (which is also linearly compact in the narrow sense). Here "matrix ring" is to mean the ring of all (possibly infinite) square matrices each of whose columns converges to zero.

The next to last section generalizes the author's previous work on pure submodules and homomorphisms [Abh. Math. Sem. Univ. Hamburg **19** (1955), 221–243; MR **16**, 899]. The last section duplicates the duality theories for linearly compact modules due to Lefschetz [Algebraic topology, Amer. Math. Soc. Colloq. Publ., v. 27, New York, 1942, pp. 78ff.; MR **4**, 84] and Kaplansky [Proc. Amer. Math. Soc. **4** (1953), 213–219; MR **14**, 720].

D. Zelinsky (Evanston, Ill.).

Divinsky, Nathan. Commutative subdirectly irreducible rings. Proc. Amer. Math. Soc. **8** (1957), 642–648.

A commutative subdirectly irreducible ring (CSI-ring) is said to be of type γ if it contains both nondivisors of zero and nilpotent elements [N. McCoy, Duke Math. J. **12** (1945), 381–387; MR **7**, 2]. The author proves that such a ring R is bound to its maximal nilideal N (i.e., $aN=0$ implies that $a \in N$). This combines with a previous result of the author [Canad. J. Math. **7** (1955), 401–410; MR **17**, 8] to yield the following generalization of a theorem due to M. Hall [Trans. Amer. Math. Soc. **48** (1940), 391–404; MR **2**, 122]: A commutative ring is isomorphic with a subdirect sum of CSI-rings, each of which is a field or is bound to its radical (Jacobson). The author investigates certain conditions which insure the existence of an identity element in a CSI-ring of type γ, and then gives an example which demonstrates that this variety of ring may lack an identity element.

W. E. Deskins (East Lansing, Mich.).

Smiley, M. F. Jordan homomorphisms and right alternative rings. Proc. Amer. Math. Soc. **8** (1957), 668–671.

Jacobson and Rickart [Trans. Amer. Math. Soc. **69** (1950), 479–502; MR **12**, 387] have called an additive mapping $a \to a'$ of an associative ring R into an associative ring R' a Jordan homomorphism if (I) $(a^2)'=(a')^2$ and (II) $(ab \cdot a)'=a'b'a'$. An arbitrary ring R in which the right multiplications a': $x \to xa$ satisfy (I) is called right alternative, and it is well known that (I) implies (II) provided $2a=0$ implies $a=0$ in R. In case R has characteristic 2, (I) may hold for right multiplications a' even though (II) does not. The reviewer [Proc. Amer. Math. Soc. **6** (1955), 716–719; MR **17**, 451] has, therefore, called a ring strongly right alternative in case both (I) and (II) hold for right multiplications. Consider now property P_0: If $x, y, z \in R$ and $(x, y, z)^2=0$, then $(x, y, z)=0$ provided x has one of the forms y, (y, z), $(y, z)y$, (y, y, z) or provided $z=wy$ and $x=(y, y, w)$. Kleinfeld [ibid. **4** (1953), 939–944; MR **15**, 595] has proved that, for right multiplications a' in a ring R, (I) and property P_0 imply that R is alternative, provided that $2a=0$ implies $a=0$. (Actually, Kleinfeld uses a slightly stronger variant of P_0, which he calls property P.) The reviewer has proved that strongly right alternative rings, having Kleinfeld's property P, are alternative, assuming $2x=0$ for all x in R. The present paper proves that (I), (II) and P_0 imply that R is alternative, utilizing methods previously developed by the author [Trans. Amer. Math. Soc. **84** (1957), 426–429; MR **18**, 715]. The paper is noteworthy in that the fracture into cases, by Kleinfeld and the reviewer, is neatly and briefly obviated.

R. L. San Soucie (Buffalo, N.Y.).

Lamprecht, Erich. Struktur und Relationen allgemeiner Gausscher Summen in endlichen Ringen. I, II. J. Reine Angew. Math. **197** (1957), 1–26, 27–48.

Let R be a finite ring, e a character of the additive group R_a of R, and Γ a representation of the multiplicative group of invertible elements of R; set $\Gamma(\xi)=0$ when ξ is a zero-divisor; set

$$\tau=\sum_{\zeta \in R}\Gamma(\xi)e(\xi);$$

τ is called the Gauss sum relative to Γ and e. The problem is to derive results concerning these Gauss sums generalizing those relative to the case where R is commutative. The author first shows that the problem may be reduced to the case where Γ is simple and R directly indecomposable; assume from now on that this is the case. The character e is called "echt" when the mapping $(\xi, \eta) \to e(\xi\eta)$ puts R_a in duality with itself (i.e., defines an isomorphism of R_a with its group of characters). The representation Γ is called "volleigentlich" when the following condition is satisfied: $\mathfrak{a}$ being an ideal $\neq\{0\}$ (left or right) of R, and $\mathfrak{M}(\mathfrak{a})$ the group of invertible elements which are $\equiv 1 \pmod{\mathfrak{a}}$, the restriction of Γ to $\mathfrak{M}(\mathfrak{a})$ does not contain the unit representation of $\mathfrak{M}(\mathfrak{a})$. The most important result is as follows: if e is "echt" and Γ "volleigentlich" then $\tau \cdot {}^t\bar{\tau}=NE$ (where ${}^t\bar{\tau}$ is the transpose conjugate of τ, N the number of elements of R and E the unit matrix). Besides, some fairly complete results are given in the case where e is not "echt" or Γ not "volleigentlich". Assuming again that e is "echt" and Γ "volleigentlich", the author proves the formulas $[\mathfrak{g}]\Lambda(\Gamma, \mathfrak{g}')=\Lambda(\Gamma^{-1}, \mathfrak{g})\tau$, $[\mathfrak{g}]\Lambda(\Gamma, \mathfrak{g}'')=\tau\Lambda(\Gamma^{-1}, \mathfrak{g})$ where the notation is as follows. $\mathfrak{g}$ is any subgroup of R_a, $[\mathfrak{g}]$ its order;

$$\Lambda(\Gamma, \mathfrak{g})=\sum_{\xi \in \mathfrak{g}}\Gamma(\xi), \quad \Lambda(\Gamma^{-1}, \mathfrak{g})=\sum_{\xi \in \mathfrak{g}}\Gamma^{-1}(\xi)$$

($\Gamma^{-1}(\xi)$ being $(\Gamma(\xi))^{-1}$ if ξ is invertible, 0 if not); $\mathfrak{g}'$ (resp. $\mathfrak{g}''$) is the set of α such that $e(\alpha\gamma)=1$ (resp. $e(\gamma\alpha)=1$) for all $\gamma \in \mathfrak{g}$. These formulas generalize formulas previously established by Schmid and Teichmüller [Abh. Math. Sem. Hansischen Univ. **15** (1943), 85–96; MR **8**, 11]. The author also generalizes to non-commutative rings the notion of Jacobi sum: let Γ and Ψ be representations of the group of invertible elements of R; then

$$J=\sum_{\xi \in R}\Gamma(\xi)\otimes\Psi(1-\xi)$$

is called the Jacobi sum relative to Γ and Ψ. Assuming that $\Gamma\otimes\Psi$ is "volleigentlich" and that there exists an "echt" character of R_a, the author expresses J in terms of certain Gauss sums.

The proof of assertion c) in lemma 2 is incorrect, and the result is actually false as can be established by an example. However this does not impair the value of the

most important results of the paper. The reader should not be afraid of the statement to the effect that the product of any complex matrix by its transpose conjugate is real (!), on p. 31, since this statement is fortunately not used in the remainder of the paper. *C. Chevalley.*

Drazin, M. P. The nilpotence of nil subrings. Amer. J. Math. **79** (1957), 67–72.

Levitzki [same J. **67** (1945), 437–442; MR **7**, 2] has shown that if a ring satisfies the maximal condition for both right and left ideals then every nil subring of this ring is actually nilpotent. The present author defines a subset of a ring to be closed if it satisfies certain conditions (any subring, for instance, is closed in his sense) which we shall not give here. He then shows that if a ring R satisfies the maximal condition on nilpotent subrings then any closed subset of the ring R generates a nilpotent subring of R. While this result is closely related to that of Levitzki it does not seem to contain it. He also shows the analogous theorem for Lie rings satisfying the maximal condition on nilpotent subrings. This is related to results of Jacobson [C. R. Acad. Sci. Paris **234** (1952), 579–581; MR **13**, 618]. *I. N. Herstein* (Ithaca, N.Y.).

McCoy, Neal H. A note on finite unions of ideals and subgroups. Proc. Amer. Math. Soc. **8** (1957), 633–637.

If $I, A_1, \cdots, A_n$ are ideals of a ring R such that I is contained in the set-theoretic union of the n ideals $A_1, \cdots, A_n$ but is not contained in any union of $n-1$ of these ideals, then it is proved that some power of I is contained in the intersection of the ideals $A_1, \cdots, A_n$. A similar result also is proved for subgroups of a group. A corollary is that if the ideal I is contained in the union of n ideals, of which at least $n-2$ are semiprime, then I is contained in some one of the ideals. *R. E. Johnson.*

Niven, Ivan; and Warren, Leroy J. A generalization of Fermat's theorem. Proc. Amer. Math. Soc. **8** (1957), 306–313.

Fermat's Theorem, namely that $a^{p-1} \equiv 1 \pmod{p}$ for any prime p and any integer a not divisible by p, may be reformulated as follows: If $f(x)$ is a polynomial with integral coefficients such that $f(x) \equiv 0 \pmod{p}$ is satisfied by every integer x, then $f(x)$ is a multiple of $x^p - x$. In the present paper it is proved that if m is any positive integer, then the set of polynomials $f(x)$ having the property that $f(x) \equiv 0 \pmod{m}$ for every integer x is an ideal with a finite set of generators. More specifically, if I/m denotes the set of integers $0, 1, 2, \cdots, m-1$ and addition and multiplication are defined modulo m, then, in the ring R of polynomials with coefficients in I/m, the subring $\mathfrak{I}(m)$ of polynomials $f(x)$ such that $f(a)=0$ for every element a in I/m is an ideal in R with a finite basis. This is proved first for the case $m=p^n$ by defining a finite set of polynomials which form a basis for the ideal $\mathfrak{I}(p^n)$ in the ring I/p^n. If $m=p_1^{n_1}\cdots p_r^{n_r}$, then the ring I/m is the direct sum of the rings $I/p_1^{n_1}, \cdots, I/p_r^{n_r}$; and the ideal $\mathfrak{I}(m)$ is a direct sum of the ideals $\mathfrak{I}(p_1^{n_1}), \cdots, \mathfrak{I}(p_r^{n_r})$, each of which has a finite basis. Finally, it is proved that the ideal $\mathfrak{I}(m)$ is principal if and only if m is a prime. *W. H. Simons* (Vancouver, B.C.).

See also: Algebras: Wolf. **Homological Algebra:** Auslander and Buchsbaum. **Algebraic Geometry:** Lagn. **Control Systems:** Moisil, Popovici, Vaidha (12t through 20th articles of the section).

Algebras

Zykov, A. A. Algebras of complexes. Mat. Sb. N.S. **41(83)** (1957), 159–176. (Russian)

The second portion of the paper is concerned with symmetric functions, the theory of representing them as combinations of functions of a simple type, and the explicit calculations involved in achieving such representation. Besides treating symmetric functions of the ordinary variety, the author discusses two-dimensional symmetric functions and symmetric functions with infinitely many arguments. The first part of the paper develops a theory for algebras of complexes, a development seemingly motivated by the desire to build a structure elaborate enough to encompass all the various result on symmetric functions as applications. The description of the elements and the operations in such an algebra is too lengthy to quote here. The theory is primarily devoted to identifying certain worthwhile systems of generators in these algebras. *R. A. Good.*

Yen, Ti. Notes on linearly compact algebras. Proc. Amer. Math. Soc. **8** (1957), 698–701.

The author proves that the Wedderburn principal theorem, $A = S \oplus N$, can be extended to linearly compact algebras A over a field K if we assume either (a) A is commutative and A/M is separably generated over K for every closed maximal ideal M of A, or (b) A/M is a total matrix algebra over K for every closed maximal ideal M. *D. Zelinsky* (Evanston, Ill.).

Albert, A. A. The norm form of a rational division algebra. Proc. Nat. Acad. Sci. U.S.A. **43** (1957), 506–509.

A finite-dimensional associative division algebra $\mathfrak{D}$ over the rational field $\mathfrak{R}$ may be written $\mathfrak{D}=(\mathfrak{Z}, S, \gamma)$, where $\mathfrak{Z}$ is a cyclic field of degree n over $\mathfrak{F}$, the center of $\mathfrak{D}$, γ is a nonzero element of $\mathfrak{F}$, and S is a generator of the cyclic automorphism group of $\mathfrak{Z}$ over $\mathfrak{F}$. Then the general element x of $\mathfrak{D}$ has the form $x=x_0+x_1y+\cdots+x_{n-1}y^{n-1}$, $x_i \in \mathfrak{Z}$, $y \in \mathfrak{D}$ with the property $y^n=\gamma$, $yz=(zS)y$. The mapping $x \to X=X_0+X_1Y+\cdots+X_{n-1}Y^{n-1}$, where $X_i=\mathrm{diag}(x_i, x_iS, \cdots, x_iS^{n-1})$ and Y is the n by n matrix (y_{rs}) with $y_{nj}=\gamma\delta_{1j}$, $y_{rs}=\delta_{r+1,s}$, is an isomorphic representation of $\mathfrak{D}$ as a subalgebra of $\mathfrak{Z}_n$. The author defines $\Delta(x)=\det X$ to be the norm form of $\mathfrak{D}$ and states that it is a homogeneous polynomial of degree n in the n^2 coordinates in $\mathfrak{F}$ of x and that it permits composition $(\Delta(xy)=\Delta(x)\Delta(y))$. Then he proves that the norm form of $\mathfrak{D}$, a division algebra of odd prime degree over its center $\mathfrak{F}$, an algebraic number field, is universal in that each nonzero element of $\mathfrak{F}$ is the norm of an element of $\mathfrak{D}$. *W. E. Deskins* (East Lansing, Mich.).

Roşculeţ, Marcel N. Algèbres d'ordre n^p, non commutatives. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **8** (1956), 501–508. (Romanian. Russian and French summaries)

L'auteur considère les algèbres d'ordre n^3, aux éléments de base $\theta_1^{\alpha}\theta_2^{\beta}\theta_3^{\gamma}$ $(\theta_1^n=\theta_2^n=\theta_3^n=1)$. La loi de multiplication de deux éléments est donnée par $\theta_1^{\alpha}\theta_2^{\beta}\theta_3^{\gamma}\cdot\theta_1^{a}\theta_2^{b}\theta_3^{c}=\rho^{p(\alpha,\beta,\gamma;a,b,c)}\theta_1^{\alpha+a}\theta_2^{\beta+b}\theta_1^{\gamma+c}$, $\rho^n=1$, φ à valeurs intégrales. On détermine la relation fonctionnelle pour la fonction φ, telle que l'algèbre soit commutative, associative, ou que, dans l'algèbre, la norme du produit de deux nombres soit égale au produit des normes. Enfin, on fait ressortir qu'une algèbre associative A, contenant A_{n^2} comme sous-

algèbre, est le produit direct $B\times A_{n^2}$, A, B et A_{n^2} ayant le même élément unité et tout élément de B étant commutatif avec tout élément de A_{n^2}.

Les résultats obtenus sont étendus aux algèbres à n^p éléments de base, construites d'une manière analogue.

Résumé de l'auteur.

★ **Wolf, Paul. Algebraische Theorie der Galoisschen Algebren. Mathematische Forschungsberichte, III.** VEB Deutscher Verlag der Wissenschaften, Berlin, 1956. xii+185 pp.

The present volume gives a general review of the algebraic theory of Galois algebras, initiated by Hasse [J. Reine Angew. Math. **187** (1949), 14–43; MR **11**, 576] as a generalization of the theory of Kummer fields. Let K be an (associative finite) algebra over a field Ω and G a finite group of automorphisms. The algebra K/Ω is called a Galois algebra with Galois group G when the group algebra $\Omega[G]$ of G over Ω is $\Omega[G]$-right isomorphic to K. With a normal basis (θ^S) $(\{S\}=G)$, whose existence is equivalent to the last stated condition for a Galois algebra, the element $w(\theta)=\sum_{T\in G}T^{-1}\theta^T$ of the group algebra $K[G]$ $(=\Omega[G]\times_\Omega K)$ is called its resolvent, and the (unique) element $C(\theta)\in\Omega[G]\times_\Omega\Omega[G]=\Omega[G\times G]$ with

$$\sum_{R,S\in G}(R^{-1}\times S^{-1})\theta^R\theta^S=\Big(\sum_{T\in G}(T^{-1}\times T^{-1})\theta^T\Big)C(\theta)$$

in the sense of the multiplication in $K[G\times G]$ is called its resolvent factor. On these notions is based the author's [Math. Nachr. **9** (1953), 201–216; MR **15**, 6; cf. also Knobloch, ibid. **6** (1951), 21–44; MR **13**, 317] foundation of the theory of Galois algebras expounded in Chap. 1 of the volume. If the Galois algebra K has a unit element and e is a primitive center-idempotent in K, then K is the direct sum $\sum_{R \bmod G^*}Ke^R$, where G^* is the subgroup of G leaving e fixed; and in case the algebra Ke over Ωe is a Galois algebra with G^* as Galois group (in natural manner), K is called kernel-Galois. It is unknown whether there is a Galois algebra with unit element that is not kernel-Galois; indeed, when K is commutative and separable, i.e., when K is an improper Galois field, K is kernel-Galois. Chap. 3 is devoted mostly to Hasse's [loc. cit.] original case where the group algebra $\Omega[G]$ is semisimple and splits, and there is a short reference to the reviewer's [J. Reine Angew. Math. **189** (1951), 100–117; MR **13**, 816] generalization to the case where the semisimplicity assumption of $\Omega[G]$ is cast off. Next, with a normal subgroup $\bar{G}$ of G, Galois algebra structure is introduced into the subalgebra $\bar{K}$ of K invariant under $\bar{G}$. Further, an improper Galois field K is considered as an improper Galois field over $\bar{K}$, where $\bar{K}$ is the invariant subalgebra of a not necessarily normal subgroup $\bar{G}$ of G assumed to be a field. In Chap. 5, Galois algebras having some fixed abelian group G as their Galois group are considered, and their semi-group with respect to the multiplication in terms of resolvent factors is defined. The last has the subgroups of separable algebras and improper fields.

In the latter half of the volume, which begins with Chap. 7, the ground field Ω is assumed to be a Galois extension, with Galois group g, of a field Ω_0. Following work of the author [Math. Nachr. **9** (1953), 281–300; **10** (1953), 233–238; J. Reine Angew. Math. **193** (1954), 166–182; MR **15**, 97, 393; **16**, 790], a necessary and sufficient condition is derived that a Galois algebra K over Ω with Galois group G have over Ω_0 an automorphism group which is a given extension of G by g. It is divided into two parts, one group-theoretical, not dealing with K, and the other algebra-theoretical. Under the assumption of the group-theoretical part, the imbedding problem – i.e., whether there is an improper Galois field K over Ω having the said property (and thus being an improper Galois field over Ω_0) – is important. A detailed study is made of the problem in the case of abelian G, including Hasse's [Math. Nachr. **1** (1948), 40–61, 213–217, 277–283; MR **10**, 426, 503] results in the case of splitting semisimple $\Omega[G]$ and Kochendörffer's [ibid. **10** (1953), 75–84; MR **15**, 282] reduction to the case G, g are p-groups. The volume closes with Beyer's [Abh. Math. Sem. Univ. Hamburg **19** (1955), 127–133; J. Reine Angew. Math. **196** (1956), 34–58; MR **19**, 8] (affirmative) solution of the imbedding problem for the case of G cyclic of odd order and $\Omega[G]$ semisimple and splitting, with a short reference to the results of Faddejew [Dokl. Akad. Nauk SSSR (N.S.) **94** (1954), 1013–1016; MR **15**, 938] and Schafarewitsch [ibid. **95** (1954), 459–461; MR **16**, 572] that here the oddness assumption of the order of G is dispensable and that the cyclic condition of G can not be weakened to the abelian. *T. Nakayama* (Nagoya).

Cartier, P. Démonstration algébrique de la formule de Hausdorff. Bull. Soc. Math. France **84** (1956), 241–249.

The Hausdorff exponential formula appears in this development as a formal identity in a free associative algebra over any field of characteristic 0. If F_n is such an algebra with generators $x_1, x_2, \cdots, x_n$ and L_n the (free) Lie algebra generated by $\{x_i\}$, then F_n can be extended to the complete direct product $\bar{F}_n$ of the homogeneous submodules of F_n. Now the exp and log mappings can be defined, and it can be proved quickly that if $e^{x_1}e^{x_2}\cdots e^{x_n}=e^z$, then $z\in\bar{L}_n$ and the homogeneous components of z can be given explicitly. The formal result follows in arbitrary associative algebras by specialization from the free algebra. Should A be a normed algebra, convergence of the expression for z is readily established. As an application of his constructions, the author derives the Birkhoff-Witt theorem specifying a standard basis for the universal associative algebra of a Lie algebra. *W. G. Lister.*

★ **Jacobson, N. Some aspects of the theory of representations of Jordan algebras.** Proceedings of the International Congress of Mathematicians, 1954, Amsterdam, vol. III, pp. 28–33. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam, 1956. $7.00.

This paper presents a concise and well written summary of Jacobson's theory of generalized representations of Jordan algebras. *A. A. Albert* (Chicago, Ill.).

Ree, Rimhak. On some simple groups defined by C. Chevalley. Trans. Amer. Math. Soc. **84** (1957), 392–400.

The reviewer has given a procedure which associates to every simple Lie algebra $\mathfrak{g}$ over the complex numbers and to any field K a group $G(\mathfrak{g}, K)$. The author proves that the groups $G(\mathfrak{g}, K)$ relative to the classical Lie algebras $\mathfrak{g}$ are those one could expect, namely the projective group of a projective space of dimension n if $\mathfrak{g}$ is of type A_n, the projective symplectic group $PSp(2n, K)$ if $\mathfrak{g}$ is of type C_n, the commutator group of the projective orthogonal group $PO(2n, K)$ if $\mathfrak{g}$ is of type D_n, the commutator group of the projective orthogonal group $PO(2n+1, K)$ if $\mathfrak{g}$ is of

type B_n and K of characteristic $\neq 2$, a subgroup of $\mathrm{Sp}(2n, K)$ (which is the full group if K is perfect) if $\mathfrak{g}$ is of type B_n and K of characteristic 2. Moreover, he proves that, when $\mathfrak{g}$ is of type G_2, $G(\mathfrak{g}, K)$ is one of the simple groups defined by Dickson. *C. Chevalley* (Paris).

See also: Fields, Rings: Smiley. **Homological Algebra:** Jans; Jans and Nakayama.

Groups and Generalizations

Karrass, A.; and Solitar, D. Note on a theorem of Schreier. Proc. Amer. Math. Soc. **8** (1957), 696–697.

Let F be a free group and let H be a finitely generated (subgroup containing a) normal subgroup of F; then H must be of finite index in F. Without the words in parentheses, the theorem is due to Schreier [Abh. Math. Sem. Hamburg. Univ. **5** (1927) 161–183]. From the generalization it follows that a subgroup H of a finitely generated free group F is of finite index in F if and only if H is finitely generated and contains, for some $d>0$, the subgroup of F generated by all dth powers of elements of F. *G. de B. Robinson* (Toronto, Ont.).

Benado, Mihail. Über die Zerlegungen einer Gruppe in direkte Faktoren. III. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **8** (1956), 5–10. (Romanian. Russian and German summaries)

[Siehe MR **17**, 1050 für Mitteilungen I, II.] Es werden hauptsächlich Sätze über verschiedene Arten von kanonischen Verfeinerungen direkter Zerlegungen des Einselements eines modularen (nicht unbedingt vollständigen) Verbandes formuliert. Beispiele:

Satz. Es besteht eine eineindeutige Zuordnung zwischen den kanonischen Verfeinerungen zweiter Art der zweigliedrigen direkten Zerlegungen (*) $\mathfrak{Z}=G_1\times G_2=H_1\times H_2$ der Σ-Gruppe $\mathfrak{Z}$ einerseits, und den kanonischen Verfeinerungen gleicher Art der von den (*) induzierten direkten Zerlegungen $Z=\mathfrak{z}(G_1)\times\mathfrak{z}(G_2)=\mathfrak{z}(H_1)\times\mathfrak{z}(H_2)$ des Σ-Zentrums Z der Gruppe $\mathfrak{Z}$, andererseits. (Betreffs der kanonischen Verfeinerungen irgendeiner Art der direkten Zerlegungen, siehe § 2 vorliegender Note.)

Satz. Besitzen je zwei zweigliedrige direkte Zerlegungen des Einselements eines modularen Verbandes kanonische Verfeinerungen erster bzw. zweiter, bzw. dritter Art, so gilt das gleiche für jeden direkten Faktor des Einselements. (Die kanonischen Verfeinerungen erster Art stimmen mit den Baerschen [Bull. Amer. Math. Soc. **54** (1948), 167–174; MR **9**, 410] überein.)

Die Beweise werden in einer anderen Zeitschrift veröffentlicht werden. *Zusammenfassung des Autors.*

Baer, Reinhold. Engelsche elemente Noetherscher Gruppen. Math. Ann. **133** (1957), 256–270.

We write $x\circ y=x^{-1}y^{-1}xy$ in a group G, and define recursively $x^{(0)}\circ y=y$, $x^{(i+1)}\circ y=x\circ[x^{(i)}\circ y]$. Then an element b is called a left-Engel element of G if, for every x of G, the sequence $b^{(i)}\circ x$ terminates in the identity, and similarly b is called a right-Engel element of G if, for every x of G, the sequence $x^{(i)}\circ b$ terminates in the identity. A number of results are derived relating the Engel elements of G to nilpotent subgroups of G. For the most part the results are restricted to Noetherian groups, as the author calls groups satisfying the maximal condition. In a Noetherian group G let $F(G)$ be the Fitting subgroup, the maximal normal nilpotent subgroup, and $H(G)$ the final term in the ascending central series. The most striking result is that $F(G)$ consists of the set of left-Engel elements of G and $H(G)$ consists of the set of right-Engel eiements of G. Since evidently $H(G)\subseteq F(G)$, it follows that every right-Engel element is a left-Engel element in a Noethe-rian group. But it is not known whether the same conclusion holds if G is not Noetherian.

Marshall Hall, Jr. (Columbus, Ohio).

Coxeter, H. S. M. Groups generated by unitary reflections of period two. Canad. J. Math. **9** (1957), 243–272.

The irreducible finite unitary reflection groups were enumerated by Shephard and Todd [same J. **6** (1954), 274–304; MR **15**, 600]. The author here takes up the case of those groups in which the reflections are all of period 2. He first shows that the three-dimensional groups generated by three reflections, other than the real orthogonal groups, are the finite groups $[111^l]^m$ defined abstractly by $R_k^2=(R_2R_3)^3=(R_3R_1)^3=(R_1R_2)^l=(R_1R_2R_3R_2)^m=E$. The groups $[111^l]^m$ and $[111^m]^l$ are isomorphic, and the group is finite whenever either l or m is 3, and for $(l, m)=(4, 4)$, $(4, 5)$ or $(5, 4)$. A simple general form for the equations of the generating reflections is given, using an affine coordinate system in which the generating reflections leave invariant the covariant coordinate planes, and the centres of the corresponding homologies are the points at infinity on the contravariant axes; here covariant and contravariant vectors are defined in relation to a hermitian form and the restrictions on l and m ensure that this is a definite form. The definitions of the groups in n-space generated by n unitary reflections of period 2 are then given, and a simple graphical notation obtained for representing these groups, and it is shown how the commutator subgroups and central quotient groups can be obtained. Next the author mentions the discrete infinite groups. He goes on to construct polytopes and honeycombs associated with the groups. One of these is of special interest in that, when the real and imaginary parts of coordinates in unitary three-space are interpreted as coordinates in Euclidean six-space, the lattice corresponding to the honeycomb is the one yielding Barnes' new extreme senary form [same J. **9** (1957), 235–242; MR **19**, 251].

J. A. Todd (Cambridge, England).

Shephard, G. C. Abstract definitions for reflection groups. Canad. J. Math. **9** (1957), 273–276.

The author gives abstract definitions for these groups, generated by reflections in unitary n-space, which require more than n reflections to generate them, thus supplementing the definitions of the groups given by the author and the reviewer [same J. **6** (1954), 274–304; MR **15**, 600] and by Coxeter [see the preceeding review].

J. A. Todd (Cambridge, England).

Suzuki, Michio. The nonexistence of a certain type of simple groups of odd order. Proc. Amer. Math. Soc. **8** (1957), 686–695.

A finite group G, such that the centralizer of any element $\neq 1$ is abelian, is said to have property W, and has been shown to be either solvable or simple by Weisner [Bull. Amer. Math. Soc. **31** (1925), 413–416]. The author shows here that a non-abelian simple W-group must have even order, and in another paper with Brauer and Wall will show that it is isomorphic with an $\mathrm{LF}(2, 2^m)$. He further shows that the alternating group of order 60 is the only Redéi group, i.e., non-abelian simple group such that

every proper subgroup of a maximal subgroup of G is abelian. His proof is based on a study of the characters of such a W-group G induced by linear characters of a maximal abelian subgroup A of order n, having a normalizer N of order nl. The 1-character ξ_0 of A induces θ_0 in N and θ_0^* in G. Another linear character ξ_t of A induces an irreducible θ_t of degree l in N and a character θ_t^* in G. There are $w=(n-1)/l$ classes C_t of G that contain elements $\sigma\neq 1$ of A, and if $w\geqq 2$ there are w characters θ_t of N to each of which corresponds a unique A-character Θ_t of G such that $\Theta_t(\sigma)=\alpha+\varepsilon\theta_t(\sigma)$ with $\varepsilon=\pm 1$, and $\Theta_t(1)=nc+\alpha+\varepsilon l$. The Θ_t are linearly independent over the C_t and equal for an element not in A. If X_j is any character of G not a Θ_t nor the 1-character, and if $X_j(\sigma)=x_j$ for some $\sigma\neq 1$ of A, then it is shown that $\theta_0^*-\theta_t^*=1-\varepsilon\Theta_t+\alpha\sum\Theta_k+\sum x_jX_j$. This character has norm $l+1$, so $\sum x_j^2\leqq l-1<(n-1)/2$ for $w\geqq 2$, and x_j is the numerically smallest residue (mod n) of the degree $X_j(1)$. If G had odd order g, then nl would be odd and $w=(n-1)/l$ even for each maximal abelian subgroup A. All characters but the principal character would be exceptional for some A_k. A succession of inequalities are then derived which lead to a contradiction; showing that g is even. *J. S. Frame* (East Lansing, Mich.).

Hewitt, Edwin; and Zuckerman, Herbert S. The irreducible representations of a semi-group related to the symmetric group. Illinois J. Math. **1** (1957), 188–213.

With the symmetric group S_n of permutations of a set S of n symbols may be associated in a natural manner (a) the semi-group T_n consisting of all n^n mappings of S upon itself, and (b) the semi-group of those mappings which are 1-1. The system (b) has been studied by W. D. Munn [Proc. Cambridge Philos. Soc. **53** (1957), 13–18; MR **18**, 465] and the irreducible representations, with their characters, determined from those of S_n.

It is impossible to go into details, but the method of procedure for determining the irreducible representations of T_n in the present paper is, like that of Munn, based on the general theory established by A. H. Clifford [Amer. J. Math. **64** (1942), 327–342; MR **4**, 4]. After giving necessary preliminary definitions and theorems the author deduces in § 4 the necessary properties of a representation (over the complex field), and in § 5 that such representations do exist. In §§ 6, 7 the construction of the irreducible representations of T_n from those of S_n is given, and this construction is illustrated in § 8 in the case $n=4$. One might conclude that the rationality of every irreducible representation of S_n implies that of T_n, but this is not explicitly stated. *G. de B. Robinson* (Toronto, Ont.).

Schützenberger, Marcel Paul. $\bar{\mathscr{D}}$ représentation des demi-groupes. C. R. Acad. Sci. Paris **244** (1957), 1994–1996.

Let $S=\{a, b, \cdots\}$ be a semigroup with a unit element. Introduce the following equivalence relations: $a\mathscr{L}b\Leftrightarrow Sa\subset Sb$; $a\mathscr{R}b\Leftrightarrow aS\subset bS$; $\bar{\mathscr{L}}=\mathscr{L}\cap\mathscr{L}^{-1}$; $\bar{\mathscr{R}}=\mathscr{R}\cap\mathscr{R}^{-1}$; $\bar{\mathscr{D}}=\bar{\mathscr{L}}\cdot\bar{\mathscr{R}}=\bar{\mathscr{R}}\cdot\bar{\mathscr{L}}$, $\bar{\mathscr{H}}=\bar{\mathscr{L}}\cap\bar{\mathscr{R}}$. Let D be a $\bar{\mathscr{D}}$-class in which we have identically: $\bar{\mathscr{L}}\cap\mathscr{R}=\mathscr{L}\cap\bar{\mathscr{R}}=\bar{\mathscr{H}}$. To each such D we can associate two representations of S by means of matrices $s\to M(s)$, $s\to N(s)$, $s\in S$. The non-zero elements of these matrices belong to a group Γ determined by D. The representation $s\to M(s)$ is equivalent to the representation of S by means of right translations if it is restricted to D. These representations are generalisations of, and related to, those given in a paper of Miller and Clifford [Trans. Amer. Math. Soc. **82** (1956), 270–280; MR **17**, 1184]. *S. Schwarz* (Bratislava).

Schützenberger, Marcel Paul. Applications des $\bar{\mathscr{D}}$ représentations à l'étude des homomorphismes des demi-groupes. C. R. Acad. Sci. Paris **244** (1957), 2219–2221.

The author shows that, under some supplementary suppositions concerning S, the homomorphisms of the semigroup S to the semigroups of matrices $\{M(s)\}$, $\{N(s)\}$ and $\{M(s)\dot{+}N(s)\}$, introduced in the paper reviewed above, are closely related to certain subsemigroups of S introduced by Dubreil [Bull. Soc. Math. France **81** (1953), 289–306; MR **15**, 680]. *S. Schwarz* (Bratislava).

See also: Partial Order, Lattices: Tamura; Martić. **Fields, Rings:** McCoy. **Structure of Matter:** Elliott, Judd and Runciman. **Quantum Mechanics:** Englert.

Homological Algebra

Auslander, Maurice; and Buchsbaum, David A. Homological dimension in local rings. Trans. Amer. Math. Soc. **85** (1957), 390–405.

Let R be a commutative ring. Besides the global dimension of R (g.dim R) (upper bound of the homological dimensions of all R-modules over R), the authors introduce the finitistic global dimension of R (f.gl.dim R) which is the upper bound of the homological dimensions of those modules whose homological dimensions are finite. Assume from now on that R is local and Noetherian, and let $\mathfrak{m}$ be its maximal prime ideal. Serre has proved that g.dim R is at least equal to the dimension of $\mathfrak{m}/\mathfrak{m}^2$ as a vector space over $R/\mathfrak{m}$; it follows that g.dim $R\geqq$dim R. The authors prove that f.gl.dim $R\leqq$dim R; in fact, they establish the relationships f.gl.dim $R=$codim R and codim $R\leqq$dim R, where codim R is the maximal length of a sequence $(x_1, \cdots, x_s)$ such that, for each $i>1$, the class of x_i is not a zero divisor in $R/\sum_{j<i}Rx_j$ and $\sum_1^s Rx_i\neq R$. Now it is clear that, if g.dim $R<\infty$, then f.gl.dim $R=$g.dim R; it follows that we then have dim $R=$dim $\mathfrak{m}/\mathfrak{m}^2$, i.e. that R is regular. It is also proved that g.dim $R<\infty$ if R is regular; thus the authors obtain a homological characterization of regular local rings. They derive various properties of regular local rings from this result: if R is regular, then so is $R_\mathfrak{p}$ for every prime ideal $\mathfrak{p}$ of R; furthermore, one has dim $R/\mathfrak{p}+$rank $\mathfrak{p}=$dim R (this holds also in any factor ring of a regular ring), and the Cohen-Macaulay theorem holds in R [these last two results have also been established by Nagata in Proc. Internat. Symposium on Algebraic Number Theory, Tokyo and Nikko, 1955, Science Council of Japan, Tokyo, 1956, pp. 191–226; MR **18**, 637].

It is proved that if R is a Noetherian local ring, E an R-module, $\hat{E}$ and $\hat{R}$ the completions of E and R, then $\hat{E}$ and $\hat{R}$ have the same homological dimension.

The authors define a ring R (not necessarily local) to be regular when the local rings of all prime ideals of R are regular; they derive various properties of these rings. *C. Chevalley* (Paris).

Jans, J. P. On segregated rings and algebras. Nagoya Math. J. **11** (1957), 1–7.

If A and B are rings (associative) and Θ: $A\to B$ a ring epimorphism, then B is said to be segregated in A if there is a subring A' of A such that Θ restricted to A' is an isomorphism. A collection $\mathscr{C}$ of rings is said to be a class of rings if it has the property that every subring of a ring in $\mathscr{C}$ is also in $\mathscr{C}$. A ring B is said to be segregated in a class

of rings $\mathfrak{C}$ if B is in $\mathfrak{C}$ and is segregated in every ring in $\mathfrak{C}$ of which it is a homomorphic image. A ring B is said to be separated in $\mathfrak{C}$ when (a) B is segregated in $\mathfrak{C}$, (b) if B is a subring of the ring D belonging to $\mathfrak{C}$ and M is a subgroup of D which is a two-sided B-module, then M is a completely reducible two-sided B-module.

Let M be a two-sided B-module. The author constructs $F(B, M)$, the free ring on B and M, as follows: $F(B, M)$ is the (weak) direct sum $B+\sum_{r=1}^{\infty} M_r$, where M_r is the tensor product over B of M with itself r times. The addition in $F(B, M)$ is the usual one for tensor products of modules, the multiplication being given by

$$b(m_{i_1}\otimes\cdots\otimes m_{i_r})=((bm_{i_1})\otimes\cdots\otimes m_{i_r}),$$

$$(m_{i_1}\otimes\cdots\otimes m_{i_r})b=(m_{i_1}\otimes\cdots\otimes(m_{i_r}b)),$$

and

$$(m_{i_1}\otimes\cdots\otimes m_{i_r})\times(m_{j_1}\otimes\cdots\otimes m_{j_s})=$$
$$(m_{i_1}\otimes\cdots\otimes m_{i_r}\otimes m_{j_1}\otimes\cdots\otimes m_{j_s}).$$

The two most striking properties of $F(B, M)$ are: (1) If B is separated in a class $\mathfrak{C}$ of rings and $F(B, M)$ belongs to $\mathfrak{C}$, then $F(B, M)$ is segregated in $\mathfrak{C}$. (2) If S is segregated in $\mathfrak{C}$, then under certain circumstances there is a subring B of S which is segregated in $\mathfrak{C}$ and a two-sided B-module M contained in S, such that if $F(B, M)$ belongs to $\mathfrak{C}$, then S is a dense subring of $F(B, M)$, where the topology of $F(B, M)$ is defined by the system of neighborhoods of (0) given by the ideals $\sum_n^{\infty} M_r$ for all n.

Specializing to the case where $\mathfrak{C}$ is the class of all finite-dimensional algebras over a fixed field, the author obtains the following results: (1) The algebra S in $\mathfrak{C}$ with radical N is segregated in $\mathfrak{C}$ if and only if S/N is separable (which is the same in this case as separated in $\mathfrak{C}$) and is isomorphic to $F(S/N, N/N^2)$; (2) if S and T are segregated algebras in $\mathfrak{C}$ with radicals N and P respectively, then S and T are isomorphic if and only if S/N^2 and T/P^2 are isomorphic. *M. Auslander* (Waltham, Mass.).

Jans, J. P.; and Nakayama, Tadasi. On the dimension of modules and algebras. VII. Algebras with finite-dimensional residue-algebras. Nagoya Math. J. **11** (1957), 67–76.

[For parts I–VI see MR **16**, 993; **17**, 453, 579; **18**, 9; **19**, 118, 14.] The principal result of this paper is the following. Let Λ be a finite-dimensional algebra over the field K with radical N. Suppose dim $(\Lambda/N^2)=n<\infty$, where $\dim(\Lambda/N^2)$ denotes the homological algebra dimension of the K-algebra Λ/N^2. Then there exists a finite-dimensional algebra Ω over the field K with radical M and an algebra epimorphism φ: $\Omega\to\Lambda$ such that $\varphi^{-1}(N^2)=M^2$ and dim $\Omega\leqq 1$. The pair (Ω, φ) is uniquely determined up to an isomorphism, and $M^{n+1}=0$ and $N^{n+1}=0$. Further, if a is a two-sided ideal of Λ, then $\dim(\Lambda/a)<\infty$, and $\dim(\Lambda/a)\leqq n$ if a is contained in N^2. If l is the number of simple components in $\Gamma=\Lambda/N$, then $n\leqq 1$. The Ω mentioned in the theorem is isomorphic to the free algebra $F(\Gamma, N/N^2)$ as defined in the above review.

M. Auslander (Waltham, Mass.).

See also: Topological Groups: Taylor.

THEORY OF NUMBERS

General Theory of Numbers

Dittmann, Gerd. Über eine Verallgemeinerung der Pythagoreischen Zahlen. Wiss. Z. Pädagog. Hochsch. Potsdam. Math.-Nat. Reihe **2** (1955/1956), 261–263.

Nicholson, A. N. A test for prime numbers. Math. Gaz. **41** (1957), 115–116.

Let

$$S_n^k(x)=x+x+\cdots+x+(2k+1)+(2k+3)+\cdots+(2n-1)$$
$$=kx+(n^2-k^2),$$

with $k<n$. Necessary and sufficient condition for $8N-1$ to be composite: there exist positive n, k $(k<n)$ such that $\frac{1}{2}N=S_n^k(\frac{1}{2})$ or $\frac{1}{2}N=S_n^k(-\frac{1}{2})$. Necessary and sufficient conditions of similar type for numbers $8N+1$, $8N\pm 3$ are given. Proofs are elementary and brief. *A. Kempner.*

Rumney, M. A simple device for testing primality. Math. Gaz. **41** (1957), 121.

An elementary method is devised which tests primality of, or factors, numbers $N=10p^2+D$, $D=1, 3, 7, 9$ (p a prime) if the primes $\leqq p$ are known, thus extending the obvious range of the standard procedure by a factor 10. The process depends upon the representation of N in the decimal system, and is based on the lemma: $10c+d$ divides $10C+D$ $(D, d=1, 3, 7, 9)$ if it divides $C-T(c)$, where $T(c)$ is linear in c depending only on d and D.

It is stated that the method could be used to advantage in an electronic computing machine. *A. Kempner.*

Roberts, J. B. On binomial coefficient residues. Canad. J. Math. **9** (1957), 363–370.

Theorems are proved on $\theta_j(n)$, the number of binomial coefficients $\binom{u}{v}$, $0\leqq v\leqq u<n$, congruent modulo prime p to j, $0\leqq j\leqq p-1$, and on $\theta(n)=\theta_1(n)+\theta_2(n)+\cdots+\theta_{p-1}(n)$.

An essential tool is a theorem by Lucas: If in the scale p, p a prime, $m=b_0+b_1p+\cdots+b_kp^k$, $n=a_0+a_1p+\cdots+a_kp^k$, then

$$\binom{n}{m}\equiv\binom{a_0}{b_0}\cdot\binom{a_1}{b_1}\cdots\binom{a_k}{b_k}\pmod{p},$$

where $\binom{r}{s}=0$ for $s>r$.

Special results obtained are the following. (1) A necessary and sufficient condition for all $\binom{u}{v}$, $0\leqq v\leqq u$, to be relatively prime to p, is that $u=(c+1)p^k-1$, $0\leqq c\leqq p-1$. (2) $\theta(cp^k+s)=\theta(cp^k)+(c+1)\theta(s)$, $0\leqq c\leqq p-1$, $1\leqq s\leqq p^k$. (3) Systems of simultaneous difference equations are established for given p; e.g. for $p=3$, $\theta(3^k)-5\theta_1(3^{k-1})-\theta_2(3^{k-1})=0$, $\theta_2(3^k)-\theta_1(3^{k-1})-5\theta_2(3^{k-1})=0$, from which $2\theta_1(3^k)=6^k+4^k$, $2\theta_2(3^k)=6^k-4^k$. (4) The final theorem establishes: $\lim \theta(n)/\theta_0(n)=0$, $n\to\infty$ (most binomial coefficients are divisible by a given prime).

A. Kempner (Boulder, Colo.).

★ **Specht, Wilhelm. Elementare Beweise der Primzahlsätze.** Hochschulbücher für Mathematik, Band 30. VEB Deutscher Verlag der Wissenschaften, Berlin, 1956. v+78 pp. DM 6.40.

Die vorliegende Monographie macht es sich zur Aufgabe, unter Voraussetzung „elementarer" Kenntnisse beim Leser eine ausführliche Darstellung der Beweise der beiden Primzahlsätze zu geben:

$$\lim_{x\to\infty}\frac{\pi(x)\log x}{x}=1,\quad \lim_{x\to\infty}\frac{\pi(k, l; x)}{x}=\frac{1}{\phi(k)}.$$

Die Beweisführung stützt sich in ihren wesentlichen Teilen auf die Arbeiten von A. Selberg und H. N. Shapiro, kombiniert ihre Gedankengänge und vereinfacht in mancher Beziehung den rechnerischen Apparat; zum Teil werden auch ältere Überlegungen von E. Landau herangezogen. *Aus der Einleitung.*

Weinert, H. J. Ein Lösungsverfahren für simultane Kongruenzen. Wiss. Z. Pädagog. Hochsch. Potsdam. Math.-Nat. Reihe **2** (1955/1956), 199–200.

Palamà, Giuseppe. Il problema di Waring. Boll. Un. Mat. Ital. (3) **12** (1957), 83–100.

An expository account, with a bibliography of 38 entries, of the salient points in the history of Waring's problem.

Stolt, Bengt. Über die diophantische Gleichung $\binom{n}{k}=Mx^m$. Arch. Math. **7** (1957), 446–449.

Let $n\geqq 2k$, $m>1$, and $k>3$. The reviewer proved [Mat. Lapok **1** (1950), 192–210; MR **13**, 208] that $\binom{n}{k}=x^m$ has no solutions in integers x, k, m and n. Let now $M=\prod p_\nu^{\alpha_\nu}$, $\alpha_\nu<m$. The author proves the existence of a $k_0=k_0(m, M)$ such that for $k>k_0$ the equation $\binom{n}{k}=Mx^m$ has no integer solutions.

[For further literature see V. Rigge, Nionde Skand. Matematiker Kongrenen, Helsingfors, 1938, Mercator, Helsingfors, 1939, pp. 155–160; Ark. Mat. Astr. Fys. **27A** (1940), no. 3; MR **2**, 145; also Erdös, Nederl. Akad. Wetensch. Proc. Ser. A. **58** (1955), 85–90; MR **16**, 797.] *P. Erdös* (Toronto, Ont.).

Barnes, E. S. The complete enumeration of extreme senary forms. Philos. Trans. Roy. Soc. London. Ser. A. **249** (1957), 461–506.

Let $\varphi=\sum a_{ij}x_ix_j$ be a positive definite quadratic form in n variables with determinant Δ, and let M be the minimum of the form for integral values of the variables. The form is said to be extreme if M^n/Δ is a local maximum for variation of the coefficients a_{ij}. If φ obtains its minimal value M for S values of the vector of variables x_i, it is said to be perfect if the form is completely defined by the values of these S minimal vectors. An extreme form is necessarily perfect.

The author makes an exhaustive study of the perfect forms in 6 variables following the methods of Voronoĭ [J. Reine Angew. Math. **133** (1907), 97–178]. Associated with a perfect form $\varphi=\sum a_{ij}x_ix_j$ with S minimal vectors $\lambda_k=\sum m_{ik}x_i$, a region R is defined in the $N=\frac{1}{2}n(n+1)$-dimensional space of the coefficients a_{ij} such that

$$\sum a_{ij}x_ix_j=\sum \rho_k\lambda_k^2 \quad (\rho_k\geqq 0;\ k=1, \cdots, S).$$

The region R is bounded by $(N-1)$-dimensional faces of the form $\sum p_{ij}a_{ij}=0$. Then for each such face, for a uniquely determined number ρ, $\varphi'(x)=\varphi(x)+\rho\sum p_{ij}x_ix_j$ is also a perfect form, called a neighbouring form of φ.

Starting with the perfect form

$$\varphi_0=\sum x_i^2+\sum_{i<j}x_ix_j,$$

the author finds all the inequivalent neighbouring forms, the neighbouring forms of these, and so on until he obtains a complete set of forms which includes equivalents of all the neighbours of these forms.

The following forms are obtained:

φ_0, $\varphi_1=\varphi_0-x_1x_2$, $\varphi_2=\varphi_0-x_1x_2-x_1x_3$,

$\varphi_3=\varphi_0-\frac{1}{2}(x_1x_2+x_3x_4+x_5x_6)$,

$\varphi_4=\varphi_0-\frac{1}{2}(x_1x_2+x_3x_4+x_3x_5+x_3x_6+x_4x_5+x_4x_6+x_5x_6)$,

$\varphi_5=\varphi_0-\frac{1}{2}(x_1x_2+x_3x_4+x_3x_5+x_4x_5)$,

$\varphi_6=\varphi_0-\frac{1}{2}(2x_1x_2+x_1x_3+x_1x_6+x_2x_5+x_4x_6+2x_5x_6)$.

Of these perfect forms φ_5 is not extreme, the others are extreme. *D. E. Littlewood* (Bangor).

Barnes, E. S. The perfect and extreme senary forms. Canad. J. Math. **9** (1957), 235–242.

The author describes the results (proved in detail in the paper reviewed above) on perfect and extreme senary forms. There are six classes of extreme forms, and one form which is perfect but not extreme. Among the classes of extreme forms is one, denoted here by ϕ_6, which has not been described before. As a representation of this form, we may take

$$\sum_1^6 x_i^2+\sum_{i<j}x_ix_j-\tfrac{1}{2}[2x_1x_2+x_1x_3+x_1x_6+x_2x_5+x_4x_6+2x_5x_6].$$

The lattice corresponding to this form has 21 minimal vectors, and its group, of order 672, is the direct product of PGL(2, 7) with a group of order 2. The form appears as a particular case ($n=6$) of the form f_n, extreme for all $n\geqq 6$, corresponding to the lattice defined by $y_1, \cdots, y_{n+1}$ which takes integral values subject to

$$\sum y_i=0, \quad \sum iy_i\equiv 0 \pmod{n+1}.$$

The author incidentally constructs an example to disprove a conjecture by Coxeter that every eutactic form in n variables with at least $\frac{1}{2}n(n+1)$ minimal vectors is perfect. *J. A. Todd* (Cambridge, England).

★ **Villaseñor Z., Francisco. El celebre teorema de Fermat y su demostración. [The celebrated theorem of Fermat and its proof.]** Mexico, D. F., 1957. v+127 pp.

Klimov, N. I. Upper estimates of some number theoretical functions. Dokl. Akad. Nauk SSSR (N.S.) **111** (1956), 16–18. (Russian)

Utilizing the sieve method of A. Selberg the author derives various estimates of prime-number-theoretic functions, which represent extensions of previously known results. Typical of these is the following theorem. Let p denote a prime, and $u_1, \cdots, u_{m-1}$ ($m\geqq 2$), k, l, integers with $(k, l)=1$. Also, let h and x denote positive reals with h fixed independently of x. $Z_h(k; u_1, \cdots, u_{m-1}; x)$ is defined as the number of primes p of the arithmetic progression $kn+l$, such that $h\leqq p\leqq h+x$, and for which $|p+u_i|$ ($i=1, \cdots, m-1$) are all prime numbers. If (a) $k=O(x^\delta)$ ($\delta<1$) and (b) $\log u_i=O(\log^c x)$ ($i=1, \cdots, m-1$), then

$$Z_h(k; u; x)\leqq \frac{m!x}{k\log^m(x/k)}\prod_p\frac{1-\omega(p)/p}{(1-1/p)^m}\left\{1+O\left(\frac{\log\log x}{\log x}\right)\right\}$$

where $\omega(p)$ denotes the number of incongruent solutions modulo p (solutions for n) of the congruence

$$(kn+l)\prod_{i=1}^{m-1}(kn+l+u_i)\equiv 0 \pmod p;$$

and the O in the above depends only on c, m, and δ. *H. N. Shapiro* (New York, N.Y.)

See also: Linear Algebra: Khan. **Fields, Rings:** Niven and Warren. **Groups and Generalizations:** Coxeter. **Analytic Theory of Numbers:** Maass; Postnikov; Gupta. **Theory of Algebraic Numbers:** Carlitz. **Lie Groups and Algebras:** Dieudonné.

Analytic Theory of Numbers

Maass, Hans. Spherical functions and quadratic forms. J. Indian Math. Soc. (N.S.) **20** (1956), 117–162.

Let $x'Tx=T[x]$ and $x'Sx=S[x]$ be two positive quadratic forms of m and n variables, respectively, with $m>n$. Let B be a subset of all m by n matrices X such that $X \in B$ implies $XV \in B$ for all non-singular matrices V. Let C be a subset of the set of all reduced positive matrices in the sense of Minkowski such that $Y \in C$ implies $\lambda Y \in C$ for λ a positive scalar. Let $a_t(B, C)$ denote the number of integral matrices G such that $G'SG=S[G]=T$, $G \in B$, $T \in C$, $|T|=t$. The author remarks that in the field of reals and under certain measure conditions on B and C an asymptotic value can be found for $A_t(B, C)=t^{-1}\sum_{r \le t} a_r(B, C)$. This paper is a contribution toward the analogous problem in algebraic number fields, based on the approximation of $\varphi(s; B, C)=\sum_{t=1}^{\infty} a_t(B, C)t^{-s}$ by a finite or infinite linear combination of certain zeta functions. Let $S=Q'Q$, $Q'=Q>0$; $f(X)=1$ or 0 according as $Q^{-1}X \in B$ or not, $g(Y)=1$ or 0 according as $Y[U] \in C$ for a unimodular U or not. It can be seen that

$$\varphi(s; B, C)=\sum_G f(QG)g(S[G])|S[G]|^{-s},$$

the summation being over a complete set of integral matrices $G=G^{(m,n)}$ of rank n such that no two differ by a unimodular right factor. The aim of the paper is to introduce a generalized class of spherical functions which are useful for the approximation of $f(X)$ for arbitrary n. These functions are too complex to be included here.

B. W. Jones (Boulder, Colo.).

Postnikov, A. G. Additive problems with a growing number of terms. Izv. Akad. Nauk SSSR. Ser. Mat. **20** (1956), 751–764. (Russian)

Let $f(x)$ be a function which assumes integral values for integral x. The author considers the number of representations of an integer N in the form $f(x_1)+\cdots+f(x_n)=N$ when $0 \le x_i \le P$, P fixed, $i=1, 2, \cdots, n$, and $n \to \infty$. An asymptotic formula is obtained for each of the cases $f(x)=x$ and $f(x)=x^2$. For example, in the case $f(x)=x$, the author shows that there exists a constant $k>1$ such that the number of solutions of the equation $x_1+x_2+\cdots+x_n=N$ in integers $0 \le x_i \le P \le k^n/n$, $i=1, 2, \cdots, n$, is given by

$$r_{n,P}(N)=\frac{(P+1)^n}{\sqrt{(\pi n(P^2+2P)/6)}}\exp\Big(-\frac{(N-nP/2)^2}{n(P^2+2P)/6}\Big)\,O\Big(\frac{(P+1)^{n-1}}{n}\Big)$$

uniformly with respect to N. The proof of the result makes use of the local limit theorem of probability theory and estimates of the trigonometric sum $\sum_{0 \le x \le P}\exp(2\pi i\alpha x)$.

W. Simons (Vancouver, B.C.).

Carleson, Lennart. Random sequences and additive number theory. Math. Scand. **4** (1956), 303–308.

A sequence S of (positive) integers is said to be a basis if every integer can be represented as the sum of a bounded number of elements of S. The standard problem of the additive theory of numbers is to decide whether a given sequence is a basis. In the present paper the author deals with a statistical version of this problem by investigating whether a sequence (chosen at random from a certain class of sequences) is 'likely' to be a basis. The main result he establishes is as follows. Let $\{\lambda_n\}_1^{\infty}$ be a sequence of positive integers such that $\log \lambda_n$ is an increasing convex function of $\log n$, and $\lim_{n \to \infty}(\log \lambda_n/\log n)<\infty$. Consider the set of all sequences $\{t_n\}_0^{\infty}$ such that $t_0=1$, $0 \le t_n < \lambda_n$ $(n \ge 1)$. With the sequence $\{t_n\}_0^{\infty}$ we associate the number

$$t=\sum_{n=1}^{\infty} t_n(\lambda_1\lambda_2\cdots\lambda_n)^{-1}$$

(which therefore belongs to the interval $0 \le t \le 1$). Then the numbers t associated with those sequences $\{t_n\}_0^{\infty}$ which are bases constitute a set of Lebesgue measure 1. The proof employs standard analytical techniques and a well-known theorem of Schnirelmann on the density of sequences [E. Landau, Über einige neue Fortschritte der additiven Zahlentheorie, Cambridge, 1937, Satz 94].

L. Mirsky (Sheffield).

Gupta, Hansraj. Partitions in terms of combinatory functions. Res. Bull. Panjab Univ. no. **94** (1956), 153–159.

The author lists two sets of formulas expressing $p(n, m)$, the number of partitions of n into at most m non-zero summands, in terms of combinatory functions, for values of $m \le 12$. *W. H. Simons* (Vancouver, B.C.).

Erdös, P. On the irrationality of certain series. Nederl. Akad. Wetensch. Proc. Ser. A. **60**=Indag. Math. **19** (1957), 212–219.

Extending previous results of the reviewer [Proc. Nat. Inst. Sci. India **13** (1947), 171–173; MR **9**, 500] the author proved [J. Indian Math. Soc. **12** (1948), 63–66; MR **10**, 594] that, for every integer $t>1$, the series $\sum_1^{\infty} d(n)/t^n$ and $\sum_1^{\infty} r(n)/t^n$ are irrational, where $d(n)$ denotes the number of divisors of n and $r(n)$ the number of solutions of $n=x^2+y^2$. He remarked that he could not prove the irrationality of the series $\sum_1^{\infty} \sigma(n)/t^n$ where $\sigma(n)$ is the sum of the divisors of n. In this paper he proves that the series $\sum_1^{\infty} 1/t^{\sigma(n)}$ is irrational. Among many interesting problems raised by the author we mention: "I do not know if a series $\sum_1^{\infty} 1/t^{n_k}$ with $\limsup(n_k/k)=\infty$ can be an algebraic number." Here $n_1<n_2<n_3<\cdots$ is a sequence of increasing positive integers. *S. Chowla.*

★ **Schneider, Theodor. Einführung in die transzendenten Zahlen.** Springer-Verlag, Berlin-Göttingen-Heidelberg, 1957. v+150 pp. DM 21.60.

This addition to the small library of modern books on trancendental numbers [see C. L. Siegel, Transcendental numbers, Princeton, 1949; MR **11**, 330; and A. O. Gel'fond, Transcendental and algebraic numbers, Gostehizdat, Moscow, 1952; MR **15**, 292] will be welcomed by mathematicians everywhere, the more so since it contains proofs of the author's fundamental work on the transcendency of elliptic and modular functions. (His even more far-reaching results on abelian functions are also mentioned, but not proved.)

In Chapter 1, the author starts with Liouville numbers, proceeds to a slight generalisation of Roth's recent theorem on the approximation of algebraic numbers, and ends with the actual construction of special classes of transcendental numbers, both Liouville and non-Liouville. Chapter 2 on the values of periodic functions begins with

a proof of the transcendency of e^{α} (special theorem of Lindemann) by means of interpolation series. Then a general theorem on the algebraic dependence of integral or meromorphic functions is proved and shown to imply the transcendency of e^{α}, of α^{β}, of $(\log \alpha)/(\log \beta)$, and in particular the author's results already referred to. Chapter 3 gives an account of the still unsatisfactory position in the classification of transcendental numbers and deals in particular with the methods due to K. Mahler and J. F. Koksma. In Chapter 4 measures of transcendency for numbers like e^{α} and α^{β} are proved and connected with the classification theory in Chapter 3. The final chapter deals with Siegel's method of proving the transcendency of solutions of linear differential equations. As examples, the general theorem of Lindemann on the exponential function and Siegel's theorem on the transcendency of Bessel functions are derived.

The book contains many valuable references to the literature. It will serve as an excellent introduction to different aspects of the theory. *K. Mahler.*

See also: General Theory of Numbers: Specht; Klimov. **Special Functions:** Myrberg.

Theory of Algebraic Numbers

Carlitz, L. Class number formulas for quadratic forms over GF[q, x]. Duke Math. J. **23** (1956), 225–235.

Let q be a power of an odd prime and let GF[q, x] be the ring of polynomials in the indeterminate x with coefficients in the Galois field of order q. G. C. Byers has studied binary quadratic forms over such rings, deriving relations between class numbers [same J. **21** (1954), 445–461; MR **17**, 350]. The author proves these results and generalizations of them by his method of singular series (Gauss sums). He concludes with a formula for the class number of GF(q)(x)($\Delta^{\frac{1}{2}}$), for $\Delta \in$ GF[q, x] and Δ not a square in the completion of this ring under the valuation with $|x|>1$. *G. Whaples* (Bloomington, Ind.).

See also: Fields, Rings: Lamprecht. **Analytic Theory of Numbers:** Maass. **Lie Groups and Algebras:** Dieudonné.

Geometry of Numbers

Descombes, Roger. Sur la répartition des sommets d'une ligne polygonale régulière non fermée. Ann. Sci. Ecole Norm. Sup. (3) **73** (1956), 283–355.

When ξ, ξ', η, η' are real and ξ, ξ' are irrational numbers, the number couples (ξ, η), (ξ', η') are defined to be equivalent if integers A, B, C, D exist so that $AD-BC=\pm 1$,

$$\xi'=\frac{A\xi+B}{C\xi+D},\quad \eta'=\frac{AD-BC}{C\xi+D}\,\eta+\frac{E\xi+F}{C\xi+D}\quad (C\xi+D>0).$$

Khintchine [Math. Ann. **111** (1935), 631–637] determined the best possible constant γ_1 such that for each couple (ξ, η) and a given ε, $\varepsilon>0$, an infinite number of integer couples (u, v) exists with $v>0$ so that $v|v\xi-u-\eta|<\gamma_1^{-1}+\varepsilon$. If (ξ, η) is not equivalent to a certain number couple (ξ_1, η_1), Cassels [ibid. **127** (1954), 288–304; MR **15**, 687] determined the best possible constant γ_2 so that infinitely many integer couples (u, v) with $v>0$ exist so that $v|v\xi-u-\eta|<\gamma_2^{-1}+\varepsilon$. The present paper gives sequences $\gamma_2, \gamma_3, \cdots$ and (ξ_2, η_2), (ξ_3, η_3), $\cdots$ with the property that if (ξ, η) is not equivalent to any of the couples (ξ_1, η_1), $\cdots$, (ξ_{r-1}, η_{r-1}), $r\geqq 2$, then γ_r is the best possible constant so that infinitely many integers couples (u, v) with $v>0$ exist for which $v|v\xi-u-\eta|<\gamma_r^{-1}+\varepsilon$. For $r>5$ the numbers γ_r, ξ_r, η_r are given by recurrence formulas.

The author shows that if (ξ, η) is not equivalent to any pair (ξ_r, η_r) $(r=1, 2, \cdots)$, then an infinity of integer pairs (u, v) with $v>0$ exists so that $v|v\xi-u-\eta|<\gamma^{-1}+\varepsilon$, γ being the limit of the sequence γ_r. There is a continuous infinity of number couples (ξ, η) for which γ is the best possible constant. The proofs depend on an algorithm defined by the use of the continued fraction development of ξ close to the one used by Cassels. They are too detailed to be described here. *D. Derry* (Vancouver, B.C.).

★ **Frank, Evelyn. Continued fractions.** A technical report, prepared under the sponsorship of the Office of Naval Research. Numerical Analysis Research, University of California, Los Angeles, Calif. 80 pp. (mimeographed)

These are lecture notes based on eight one-hour lectures. They were designed to acquaint the listeners with some of the topics considered in continued fraction theory, with particular emphasis on those topics which would be useful in problems of numerical analysis. These notes are in no way to be construed as a complete theory of the subject, but merely as an introduction to the theory and a sample of the types of problems involved. The topics considered are: (1) definitions and basic formulas, (2) equivalent and corresponding continued fractions, various types of continued fraction expansions, (3) convergence and divergence of continued fractions, (4) the approximation error, (5) applications to problems of stability. *From the author's summary.*

See also: Analytic Theory of Numbers: Schneider.

ANALYSIS

Functions of Real Variables

★ **Miller, Kenneth S. Advanced real calculus.** Harper & Brothers, New York, 1957. viii+185 pp. $5.00.

This book is 182 pages of carefully written analysis. It deals with the Fundamentals of the Calculus. It uses the Dedekind cut of the rational numbers to define the irrational numbers. This leads to an adequate foundation on which to place the concepts of bounds, convergence, limit, limit points of sets of real numbers, Bolzano-Weierstrass and Heine-Borel theorems. In Chapter III the definitions and results of the first two chapters are used to develop the properties of a function of a single real variable, including its derivative and differential. One is pleased to see that the symbols used for the derivative are $f'(x)$, $df(x)/dx$, and one wonders why the frequently used symbol $D_x y$ is not mentioned. This satisfaction is turned to grief when a few pages on one finds dy/dx noted as a symbol for the derivative. Its mention here serves no purpose, at any rate no useful purpose. Likewise, in the following lines the term infinitesimal is superfluous. Surely there is neither cause for confusion nor lack of rigour in defining the differential dy as $f'(x)$ multiplied by the number dx. Then, if $dy=f'(x)dx$, $f'(x)$ is represented by the ratio of the two numbers, differential of y and the number dx; $f'(x)=dy/dx$.

Chapter IV gives a complete treatment of the Riemann integral. The definitions as the common limit of the upper and lower Darboux sums and as the limit of sums of the form $\sum f(\xi_i)(x_i-x_{i-1})$ are both given and discussed in detail. The properties are stated and proved in detail. It is stated that $\int_a^\infty f(x)dx$ exists if there exists a number I such that given ε there exists X for which $|\int_a^x f(x)dx-I|<\varepsilon$ when $x>X$. Then $\int_{-\infty}^a f(x)dx$ is defined as the limit of $\int_b^a f(x)dx$ as $b\to-\infty$. Why the two types of definition when either is adequate for both cases? In all the proofs, including those for the necessary and sufficient conditions for both proper and improper integrals, the full, detailed ε, δ-treatment is given. Furthermore this is done in a way that does not become tiresome, an achievement for which the author deserves much credit. The same may be said for the work on series and sequences and on repeated and double integrals in Chapters V, VI and VII. In Chapter V the usual theorems on the convergence of sequences of functions are given, with examples to illustrate unusual behaviour. Chapters VI and VII deal with functions of several real variables and multiple integrals. Emphasis is put on functions of two variables. The Heine-Borel theorem is proved for plane sets. The repeated integral of a function $f(x, y)$ is defined, with finite and infinite limits of integration, and conditions are obtained for the equality of the repeated integrals. The double integral is then defined and sufficient conditions are given for equality of the double integral and either of the repeated integrals.

At the end of each chapter there is a substantial list of carefully chosen problems, the working of which will lead to a better understanding of the theory and a further knowledge of useful results. *R. L. Jeffery.*

Bečvář, Jiří. Sur les fonctions monotones continues dont les représentations graphiques possèdent une longueur maximale. Časopis Pěst. Mat. **81** (1956), 172–181. (Czech. Russian and French summaries)

The paper is concerned with the existence of non-decreasing continuous functions on an interval (a, b), whose graph has maximal length $f(b)-f(a)+b-a$. The author shows that these functions are dense, with respect to the metric $\|f-g\|=\max|f(x)-g(x)|$, in the set of non-decreasing functions with fixed $f(a)$ and $f(b)$.

F. Wolf.

★ **Bonferroni, Carlo Emilio. La mediana ponderata in una distribuzione continua.** Scritti matematici in onore di Filippo Sibirani, pp. 21–31. Cesare Zuffi, Bologna, 1957.

Data una funzione reale $x=x(t)$, continua per $a\leq t\leq b$, siano λ e Λ il minimo e rispettivamente il massimo di essa in (a, b). Per un qualunque y tale che $\lambda\leq y\leq\Lambda$, indichiamo con $I(x\leq y)$ e con $I(x=y)$ gli insiemi dei valori t di (a, b), tali che $x(t)\leq y$, rispettivamente $x(t)=y$. Indi poniamo $m(y)$=misura di $I(x\leq y)$, $\bar{m}(y)$=misura di $I(x=y)$, poi $x_\theta(t)=y$ per tutti i t tali che $a+m(y)-\bar{m}(y)\leq t\leq a+m(y)$. Rimane così definita in (a, b), facendo assumere alla y tutti i valori tali che $\lambda\leq y\leq\Lambda$, una funzione $x_\theta(t)$ non decrescente, che l'Autore chiama la "commutazione crescente" della $x(t)$ in (a, b). Analogamente l'Autore definisce la "commutazione decrescente" della $x(t)$. Studia poi talune proprietà interessanti delle commutazioni e ne fa applicazione sia alle medie ponderate, sia alle mediane ponderate. Il recensore deve però osservare che il lavoro è purtroppo insufficiente in più punti: per es. al n. 4 ove la definizione data delle due commutazioni non appare accettabile per funzioni discontinue. *T. Viola* (Roma).

★ **Chisini, Oscar. Alcuni teoremi sulle medie.** Scritti matematici in onore di Filippo Sibirani, pp. 81–86. Cesare Zuffi, Bologna, 1957.

Nel Period. Mat. (4) **9** (1929), 106–116 l'Autore indicò genericamente con media di un gruppo di n valori reali $x_1, x_2, \cdots, x_n$, relativamente ad una prefissata funzione reale $y=f(x_1, x_2, \cdots, x_n)$, quel valore $\bar{x}$ tale che $f(x_1, x_2, \cdots, x_n)=f(\bar{x}, \bar{x}, \cdots, \bar{x})$, ammettendo ovviamente l'esistenza e l'unicità di esso. Tale definizione, cui l'Autore aggiunse qualche nuova idea complementare [Schola et Vita **5** (1930), 181–190] e di cui altri già fecero un primo studio [P. Martinotti, Giorn. Econ. Riv. Statist. (4) **71** (1931), 291–303; B. de Finetti, Giorn. Ist. Ital. Attuari **2** (1931), 369–396; H. Jecklin, Vierteljschr. Naturforsch. Ges. Zürich **93** (1948), 35–41; Metron **15** (1949), 3–11; Comment. Math. Helv. **22** (1949), 260–270; MR **9**, 502; **11**, 503; **10**, 685], viene qui ripresa limitatamente al caso che i valori x_i di cui si vuole la media siano molto vicini alla media stessa, e quindi anche fra loro. In tal caso $\bar{x}$ è detto "misura media" o, brevemente, "misura" dei valori x_i, e l'Autore fa la convenzione che, qualora si sviluppi la $f(x_1, x_2, \cdots, x_n)$, supposta analitica, in serie di Taylor nell'intorno delle $x_i=\bar{x}$, siano trascurabili i termini in cui le differenze $(x_i-\bar{x})$ figurano con gradi complessivi superiori al primo. Una tale convenzione, osserva l'Autore, dipende "dalla questione che si considera e dall'ordine d'approssimazione che essa comporta."

L'Autore dimostra anzitutto due teoremi molto significativi ed interessanti: 1) Per una funzione $f(x_1, x_2, \cdots, x_n)$ nelle condizioni dette, la misura delle x_i è la media aritmetica ponderata (*) $\bar{x}=\sum_i p_i x_i/\sum_i p_i$ che ha per peso p_i, relativo a ciascuna x_i, la corrispondente f_{x_i} calcolata dando a tutte le x_i l'unico valore $\bar{x}$; 2) Come pesi delle x_i possono, nella formula (*), essere assunte le derivate f_{x_i} calcolate non per $x_i=\bar{x}$, ma per i valori, di cui si vuole la media, che assumono queste variabili nel caso in esame. L'Autore deduce poi due corollari e termina con alcune osservazioni critiche sul concetto di "valore vero", le quali sembrano giustificare molto validamente la definizione da lui data. *T. Viola* (Roma).

Markus, S. On functions continuous in each variable. Dokl. Akad. Nauk SSSR (N.S.) **112** (1957), 812–814. (Russian)

The author extends to n variables the unicity theorem for a function $f(x, y)$ continuous in each variable in a domain G, which is given in a dense subset E of G. In the case in which f is supposed to vanish in E and continuity in each variable is only assumed in $G-E$, the paper corrects an earlier one by the same author [Com. Acad. R. P. Romîne **5** (1955), 1563–1568; MR **17**, 1064].

L. C. Young (Madison, Wis.).

Rosenstock, Herbert B. Critical points in three dimensions. Phys. and Chem. Solids **2** (1957), 44–54.

Given a function of three variables, which we assume differentiable for simplicity, its critical points are the points where all first derivatives simultaneously vanish. Critical points are either extrema (maxima or minima) or saddle points. Their determination is of importance for various problems in solid state physics. The present paper determines what information can be obtained concerning the critical points inside a three-dimensional region from the knowledge of the critical points of the function on the two-dimensional boundary of the region. The obtainable information turns out to be the difference between the

number of extrema and the number of saddle points. A method to obtain upper bounds for the sum of these two numbers is discussed but it does not lead to results of practical interest. *L. Van Hove* (Utrecht).

Macbeath, A. M. A criterion for differentiability. Edinburgh Math. Notes no. **40** (1956), 8–11.

Cauchy's condition for convergence gives a necessary and sufficient condition for the convergence of a sequence without explicitly mentioning the limit. Can there be given necessary and sufficient conditions for the differentiability of a function of n variables without explicitly mentioning the partial derivatives? The author answers this question in the affirmative. Consider the case of one variable: A necessary and sufficient condition for $f(x)$ to be differentiable at $x=0$ is that for $\varepsilon>0$ there exists $\delta>0$ such that $|f(x)/x-f(y)/y|<\varepsilon$ when $|x|, |y|<\delta$. This condition can be written in the determinant form

$$\begin{vmatrix} f(x) & f(y) \\ x & y \end{vmatrix} < \varepsilon|x|\cdot|y|,$$

and in this form can be generalized immediately.

Let x denote the set of n variables $(x^1, x^2, \cdots, x^n)$; x_k is a vector $(x_k^1, x_k^2, \cdots, x_k^n)$; $|x|=[\sum(x^i)^2]^{\frac{1}{2}}$. A function $f(x)$ with $f(0)=0$ is differentiable at $x=0$ if there is a linear function $l(x)=p_1x^1+\cdots+p_nx^n$ such that $f(x)=l(x)+O(|x|)$, $|x|\to 0$. Let $x_1, \cdots, x_{n+1}$ be any $n+1$ vectors, their components making up $n(n+1)$ variables in all. Let $f(x)$ be any function of n variables with $f(0)=0$. Define $D(f; x_1, \cdots, x_{n+1})$ to be the $(n+1)\times(n+1)$ determinant whose rth row is $f(x_r), x_r^1, \cdots, x_r^n$. A necessary and sufficient condition that $f(x)$ be differentiable at $x=0$ is that given $\varepsilon>0$ there is $\delta>0$ such that

$$|D(f; x_1, \cdots, x_{n+1})| < \varepsilon|x_1|\cdots|x_{n+1}|$$

when $|x_1|, \cdots, |x_{n+1}|<\delta$.

R. L. Jeffery (Kingston, Ont.).

Martin, Abram V. A note on derivatives and neighborly functions. Proc. Amer. Math. Soc. **8** (1957), 465–467.

The concept of 'neighborliness' was introduced by W. W. Bledsoe [same Proc. **3** (1952), 114–115; MR **13**, 634] as a generalization of continuity. The present author proves that if F is a real-valued function of a real variable and if the derivative F' exists everywhere and is Riemann integrable over every finite interval, then F' is neighborly. He points out that the hypothesis of Riemann integrability cannot be discarded, and also that a function which is Riemann integrable and neighborly need not be a derivative. *U. S. Haslam-Jones.*

Džrbašyan, M. M. On a quasi-analytic class of functions of weighted polynomial approximation on the entire real axis. Akad. Nauk Armyan. SSR. Dokl. **23** (1956), 97–102. (Russian. Armenian summary)

Let $p(x)>0$ be a continuous even function defined on $(-\infty, +\infty)$ which is monotone increasing for $x>0$ and satisfies $\lim_{x\to+\infty} x^n e^{-p(x)}=0$. Denote by $C[p(x)]$ the set of all continuous $f(x)$ on $(-\infty, +\infty)$ which satisfy $\lim_{|x|\to\infty} f(x)e^{-p(x)}=0$ and put

$$E_n(f, p)=\inf_{Q_n}\left\{\max_{-\infty<x<+\infty}|f(x)-Q_n(x)|e^{-p(x)}\right\},$$

where Q_n ranges over all polynomials of degree $\leqq n$. For a sequence $n_1<n_2<\cdots$ of natural integers n_k define $C_{n_k}[p(x)]$ as the class of all $f(x)\in C[p(x)]$ satisfying $E_{n_k}(f, p)\leqq K\exp(-\gamma f_1^{n_k} q(y)^{-1}dy)$ $(k=1, 2, \cdots)$ under the condition $\int_1^\infty p(x)x^{-2}dx=\infty$, where $K>0$, $\gamma>0$ are constants independent of k and $q(y)$ is the inverse of $p(x)$. Then the functions of $C_{n_k}[p(x)]$ are determined on the entire real axis if their values are known in an arbitrary real interval. *L. Fuchs* (Budapest).

See also: Combinatorial Analysis: Perfect. **Measure, Integration:** Neugebauer; Cesari. **Functions of Complex Variables:** Nedelcu; Yûjôbô. **Harmonic Functions, Convex Functions:** Górski. **Trigonometric Series and Integrals:** Kestelman. **Integral Transforms:** du Plessis (two articles).

Measure, Integration

Gladkiĭ, A. V. On the relationship between descriptive measurability, absolute measurability and the Baire property. Mat. Sb. N.S. **41(83)** (1957), 3–6. (Russian)

Le but de cette note est de démontrer, en se servant de l'hypothèse du continu, l'existence d'un ensemble ayant la propriété de Baire qui est mesurable absolument, mais non mesurable déscriptivement. Les ensembles de Lusin [voir W. Sierpiński, Hypothèse du continu, Warszawa-Lwów, 1934] jouent un rôle important dans la démonstration de cette proposition. *M. Kondô.*

Gavrilyuk, V. T. Certain questions of convergence of many-dimensional singular integrals. Dopovidi Akad. Nauk Ukrain. RSR **1956**, 523–526. (Ukrainian. Russian summary)

Certain theorems on conditions for the convergence of one-dimensional improper integrals are extended to many-dimensional improper integrals. The convergence is considered at generalized Lebesgue points of the generating function. Certain properties of multi-dimensional classical integrals of Poisson and Fejer are also derived. *H. P. Thielman* (Ames, Ia.).

Taylor, S. J. An integral of Perron's type defined with the help of trigonometric series. Quart. J. Math., Oxford Ser. (2) **6** (1955), 255–274.

Sei $\Lambda(x)$ eine nach Lebesgue integrierbare Funktion mit der Periode 2π, und sei

$$\Lambda(r, x)=\frac{1}{\pi}\int_0^{2\pi}\Lambda(x+t)\,\frac{1-r^2}{1-2r\cos t+r^2}\,dt.$$

Dann wird $\Lambda(x)$ als Abel-stetig an einer Stelle x_0 bezeichnet, wenn $\Lambda(r, x_0)\to\Lambda(x_0)$ für $r\to 1$; ferner werden die verallgemeinerten zweiten Abel-Derivierten eingeführt:

$$\underline{AD}^2\Lambda(x)=\liminf_{r\to 1}\partial^2\Lambda(r, x)/\partial x^2,$$

$$\overline{AD}^2\Lambda(x)=\limsup_{r\to 1}\partial^2\Lambda(r, x)/\partial x^2.$$

Nunmehr gewinnt Verf. einen neuen Integralbegriff, bei dem er das Integrationsintervall durch lineare Transformation in $(0, 2\pi)$ übergeführt denkt, auf folgende Weise: Sei $f(x)$ in $(0, 2\pi)$ fast überall definiert und periodisch fortgesetzt. Eine Konstante M und eine Funktion $\Phi(x)$ werden dann als ein oberes Approximationspaar bezeichnet, wenn 1) $\Lambda(x)=\Phi(x)-Mx^2/4\pi$ periodisch mit der Periode 2π ist, 2) $\Lambda(x)$ nach Lebesgue integrierbar und überall Abel-stetig ist, 3) $\Lambda(x)$ approximativ-stetig und so beschaffen ist, daß jede perfekte Menge aus $(0, 2\pi)$ eine nicht leere Teilmenge enthält, in der $\Lambda(x)$ nach oben

halbstetig ist, 4) $\Phi(-2\pi)=\Phi(2\pi)=0$ ist, 5) $\underline{AD}^2\Phi(x)\geqq f(x)$ p. p. und mit höchstens abzählbar vielen Ausnahmestellen auch $\underline{AD}^2\Phi(x)>-\infty$ ist, 6) $\lim_{r\to 1}(1-r)\partial^2\Lambda(r, x)/\partial x^2=0$ ist. Es wird nun für die Menge aller oberen Approximationspaare das inf M gebildet und wird bewiesen, daß dieses dasselbe bleibt, wenn man für $\Phi(x)$ bloß solche Funktionen zuläßt, welche keine oben bei 5) zulässigen Ausnahmestellen haben. In analoger Weise werden untere Approximationspaare m, $\varphi(x)$ betrachtet, und wird das sup m gebildet. Dann wird gezeigt, daß stets $M\geqq m$ und folglich auch inf $M\geqq\sup m$ ist. Falls nun hier Gleichheit stattfindet, dann wird $f(x)$ als AP-integrabel bezeichnet und wird geschrieben:

$$\text{AP-}\int_0^{2\pi} f(x)dx=\inf M=\sup m.$$

Die bis zur Gewinnung dieser Definition erforderlichen Beweise werden vielfach durch Zurückführung auf genau angegebene Literaturstellen geführt, beanspruchen aber doch schon etwa die Hälfte der Arbeit. In der zweiten Hälfte werden dann einige erwünschte Eigenschaften des AP-Integrals nachgewiesen, daß es z.B. distributiv ist und daß es für zwei fast überall gleiche Funktionen denselben Wert hat. Vor allem wird gezeigt, daß eine nach Perron integrierbare Funktion stets auch zum selben Wert AP-integrierbar ist. Ferner, daß für überall nichtnegative Funktionen die Tragweite des AP-Integrals (ebenso, wie die des Perron-Integrals) nicht über Lebesgue hinausgeht. Weiterhin folgen noch einige Anwendungen auf die Integration trigonometrischer Reihen.

O. Perron (Zbl. **66** (1957), 300).

Nakanishi, Shizu. L'intégrale de Denjoy et l'intégration au moyen des espaces rangés. I. Proc. Japan Acad. **32** (1956), 678–683.

Pour qu'une fonction numérique réelle $f(x)$ définie sur un intervalle réel soit intégrable au sens de Denjoy, il faut et il suffit qu'il existe dans l'espace rangé ε [défini par K. Kunugi, Proc. Japan Acad. **32** (1956), 215–220; MR **18**, 567] une suite fondamentale jouissant de certaines propriétés. L'A. s'appuie sur des résultats de S. Enomoto [Osaka Math. J. **7** (1955), 69–102; MR **17**, 246].

A. Appert (Angers).

Mařík, Jan. La représentation d'une fonctionelle par une intégrale. Czechoslovak Math. J. **5(80)** (1955), 467–487. (Russian. French summary)

Let P be any non-void set, and $\mathfrak{Z}$ a linear space of real-valued functions defined on P such that $f\in\mathfrak{Z}$ implies $|f|\in\mathfrak{Z}$ and $\min(f, 1)\in\mathfrak{Z}$. Let J be a non-negative linear functional defined on $\mathfrak{Z}$ such that (*) $f_n\in\mathfrak{Z}$, $f_1\geqq f_2\geqq\cdots$ and $\lim_{n\to\infty} f_n(p)=0$ for all $p\in P$ imply that $\lim_{n\to\infty} J(f_n)=0$. Many writers, beginning with P. J. Daniell [Ann. of Math. (2) **19** (1918), 279–294] have studied the representability of J by an integral with respect to a countably additive measure μ on P such that all f in $\mathfrak{Z}$ are μ-measurable [see, e.g., Loomis, Amer. J. Math. **76** (1954), 168–182; MR **15**, 631; Glicksberg, Duke Math. J. **19** (1952), 253–261; MR **14**, 288; M. H. Stone, Proc. Nat. Acad. Sci. U.S.A. **34** (1948), 336–342, 447–455, 483–490; **35** (1949), 50–58; MR **10**, 24, 107, 239, 308; and the reviewer, Ark. Mat. **2** (1952), 269–282; MR **14**, 545]. The author first sketches the construction of the measure μ for which $J(f)=\int_P f(p)d\mu(p)$. He next gives various conditions under which every non-negative linear functional J on $\mathfrak{Z}$ will satisfy condition (*). For a real-valued function f on P, let $N_f=\{p: p\in P, f(p)\neq 0\}$. If $\mathfrak{M}$ and $\mathfrak{N}$ are linear spaces of functions on P, $\mathfrak{N}\subset\mathfrak{M}$, and if $f\in\mathfrak{M}$, $g\in\mathfrak{N}$, and $N_f\subset N_g$ implies $f\in\mathfrak{N}$, then $\mathfrak{N}$ is said to be a normal part of $\mathfrak{M}$. Theorem: Let $\mathscr{A}$ be a σ-algebra of subsets of P, and let $\mathfrak{Z}$ be a normal part of the space of all finite-valued $\mathscr{A}$-measurable functions of P. Then every non-negative linear functional on $\mathfrak{Z}$ has the property (*). For the case in which $\mathscr{A}$ is all subsets of P, this theorem is due to G. W. Mackey [Bull. Amer. Math. Soc. **50** (1944), 719–722; MR **6**, 70]. Theorem: Let P be a topological space and let $\mathfrak{Z}$ be a normal part of the space of all continuous real-valued functions on P. Then every non-negative linear functional on $\mathfrak{Z}$ has the property (*). For the case in which $\mathfrak{Z}$ is all continuous functions on P, this theorem is due to the reviewer [Fund. Math. **37** (1950), 161–189; MR **13**, 147]. Several other sufficient conditions for (*) always to hold are also given.

E. Hewitt.

Neugebauer, Christoph J. A cyclic additivity theorem of a functional. I. Riv. Mat. Univ. Parma **7** (1956), 33–49.

Let $\mathfrak{P}$ be a collection of Peano spaces lying in a metric space M, let $\mathfrak{A}_0$ be the class of all subsets of M which are the unions of a finite number of disjoint Peano spaces in $\mathfrak{P}$, and let $\mathfrak{A}$ consist of all subsets A of M which are either in $\mathfrak{A}_0$ or satisfy the condition that there is a sequence $Q_n\in\mathfrak{A}_0$, $Q_n\subset A^\circ$ (the interior of A), $n=1, 2, \cdots$, such that for any compact set $K\subset A^\circ$ there is an integer n_0 such that $K\subset Q_n\subset A^\circ$ for $n\geqq n_0$. For a second metric space P^* let $(\mathfrak{T}, \mathfrak{A})$ be the class of all continuous mappings (T, A) from a set $A\in\mathfrak{A}$ into P^*. An unrestricted factorization of (T, A) consists of a Peano space $\mathfrak{M}$, a subset $\mathfrak{M}^*$ of $\mathfrak{M}$, and two continuous mappings, f from A into $\mathfrak{M}^*$ and s from $\mathfrak{M}^*$ into P^*, such that $T=sf$. We write $T=sf$, $f: A\to\mathfrak{M}^*$, $s: \mathfrak{M}^*\to P^*$, $\mathfrak{M}^*\subset\mathfrak{M}$. For a proper cyclic element C of $\mathfrak{M}$ let r_C be the unique continuous monotone retraction of $\mathfrak{M}$ onto C. Let $\Phi(T, A)$ be a functional defined for each $(T, A)\in(\mathfrak{T}, \mathfrak{A})$ and satisfying the following conditions: (α) $\Phi(T, A)$ is real-valued and non-negative (possibly $+\infty$); (β) if A is a finite union of disjoint Peano spaces $P_1, \cdots, P_n$ in $\mathfrak{P}$ and $(T, A)\in(\mathfrak{T}, \mathfrak{A})$, then $\Phi(T, A)=\Phi(T, P_1)+\cdots+\Phi(T, P_n)$; ($\gamma$) if $A\in\mathfrak{A}-\mathfrak{A}_0$, Q_n ($n=1, 2, \cdots$) is the sequence of Peano spaces in $\mathfrak{P}$ defined above and $(T, A)\in(\mathfrak{T}, \mathfrak{A})$, then $\Phi(T, A)=\lim\Phi(T, Q_n)$ for $n\to\infty$; (δ) for two sets A', A'' in $\mathfrak{A}$ with $A'\subset A''$ and (T, A'), (T, A'') in $(\mathfrak{T}, \mathfrak{A})$ we have $\Phi(T, A')\leqq\Phi(T, A'')$; ($\varepsilon$) for $P\in\mathfrak{P}$, $(T, P)\in(\mathfrak{T}, \mathfrak{A})$ and an unrestricted factorization $T=sf$, $f: P\to\mathfrak{M}$, $s: \mathfrak{M}\to P^*$, we have $\Phi(T, P)=\sum\Phi(sr_C f, P)$, $C\subset\mathfrak{M}$.

The main result of this paper is the following theorem which is a generalization of a result of Mickle and Radó [Trans. Amer. Math. Soc. **66** (1949), 347–365; MR **11**, 167]. Let $T=sf$, $f: A\to\mathfrak{M}^*$, $s: \mathfrak{M}^*\to P^*$, $\mathfrak{M}^*\subset\mathfrak{M}$ be an unrestricted factorization of $(T, A)\in(\mathfrak{T}, \mathfrak{A})$ and $\Phi(T, A)$ be a functional defined for each $(T, A)\in(\mathfrak{T}, \mathfrak{A})$ and satisfying the above conditions (α)–(ε). Then $\Phi(T, A)=\sum\Phi(sr_C f, G_C)$, $C\in\mathfrak{R}$, where $\mathfrak{R}$ is the class of proper cyclic elements C of $\mathfrak{M}$ for which there is at least one component G of A such that $r_C f(G)\subset M^*$ and $G\cap A^\circ\neq\emptyset$ if $A\in\mathfrak{A}-\mathfrak{A}_0$, and where, for each $C\in\mathfrak{R}$, G_C is the union of all such components G.

E. J. Mickle

★ **Cesari, Lamberto. Retraction, homotopy, integral.** Proceedings of the International Congress of Mathematicians, 1954, Amsterdam, vol. III, pp. 77–84. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam, 1956. $7.00.

This is an excellent expository paper on surface area

theory and its application in various fields of analysis. The term "surface" is interpreted as a continuous mapping T from a "two-dimensional" set A into Euclidean three-space E_3. For definiteness we assume (as does the author in the greater part of the paper) that A is a simply connected Jordan region in a Euclidean plane. If T is continuously differentiable, then its "area" is given by the classical area-integral (the double integral of the square root of the sum of the squares of the three Jacobians corresponding to T). For this smooth case, one finds in classical analysis an extensive array of results involving surface integrals (transformation formulas, contour integration formulas, and a large body of theorems in classical differential geometry and calculus of variations). The task of extending these classical results in conformity with the requirements of modern analysis is a prodigious one, and in spite of the great advances accomplished in the course of the past several decades, many fundamental problems still await solution. The present paper furnishes a valuable account of many of the principal lines of research in this general field, with particular emphasis upon topics where the author and his associates made outstanding contributions. These include concepts of bounded variation, absolute continuity, and generalized Jacobians for continuous mappings T, and the principal theorems relating to these concepts; tangential properties of general continuous surfaces; the theory of a general Weierstrass-type double integral; extremely general forms of the Gauss-Green and Stokes theorems; theorems on homotopy and retraction for general continuous surfaces; and far-reaching results in calculus of variations for double integrals.

T. Radó (Columbus, Ohio).

See also: Harmonic Functions, Convex Functions: Rauch; Smith. **Trigonometric Series and Integrals:** Kestelman; Upton. **Topological Vector Spaces:** Gel'fand. **Banach spaces, Banach Algebras, Hilbert Spaces:** Umegaki. **Differential Geometry:** Pogorelov; Whitney. **statistics** Adhikari and Joshi.

Functions of Complex Variables

Trohimčuk, Yu. Yu. On two problems of N. N. Luzin. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 5(71), 215–222. (Russian)

Among the notes of the late N. N. Lusin [cf. V. S. Fëdorov, Uspehi Mat. Nauk (N.S.) **7** (1952), no. 2(48), 7–16; MR **13**, 810] was the question of the structure of the set $\mathfrak{M}_x$ of monogenicity of a function $f(z)$ continuous in a domain D; the set $\mathfrak{M}_x$ is defined to be the intersection $\bigcap_{\varepsilon>0} M_\varepsilon$, where M_ε is the set of all numbers

$$[f(z+\Delta z)-f(z)]/\Delta z$$

with $z+\Delta z \in D$ and $0<|\Delta z|\leqq\varepsilon$. Talmonov [ibid. **8** (1953), no. 5(57), 169–171; MR **15**, 612] showed, for example, that $\mathfrak{M}_x$ can fill a circle. The author considers those additional properties to be obtained if some hypothesis concerning the partial derivatives of $f(z)=u+iv$ is imposed, and shows that $\mathfrak{M}_x$ can be a point, the circumference of a circle, or the whole plane. Furthermore, if the set $\mathfrak{M}_x$ does not depend on z, then either $\mathfrak{M}_x$ is the whole plane or else $f(z)$ is a function of the form $Az+B\bar{z}+C$, where A, B, C are constants.

A. J. Lohwater.

Nedelcu, Mariana. Formules de médiation pour le polynôme aréolaire d'ordre n, dans l'espace à trois dimensions. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **8** (1956), 51–58. (Romanian. Russian and French summaries)

For the definition of areolar polynomials of order n see a previous paper by the author [Acad. R.P. Romîne. Stud. Cerc. Mat. **7** (1956), 197–237; MR **18**, 646]. In the present paper the author gives further representations of areolar polynomials in powers of r (radial coordinate) and mean values over spheres. *A. Erdélyi.*

★ **Ricci, Giovanni. Sul resto delle serie di potenze alla periferia del cerchio di convergenza.** Scritti matematici in onore di Filippo Sibirani, pp. 233–242. Cesare Zuffi, Bologna, 1957.

Let $f(z)=\sum a_k z^k$ have radius of convergence one and set $f_n(z)=\sum_{k=0}^{n} a_k z^k$. Let $\zeta=e^{i\gamma}$ and set $\phi(\zeta)=\lim_{z\to\zeta,|z|<1} f(z)$ when this limit exists. In particular $\phi(\zeta)=f(\zeta)$ in all regular points of the circumference. In an earlier paper [Ann. Scuola Norm. Sup. Pisa (3) **8** (1954), 121–131; MR **17**, 23] the author concerned himself with determining the order of $|f_n(\zeta)-f(\zeta)|$ near a regular point when $a_n=O(n^{-\gamma})$. A particular result included the result of Fatou and Riesz [Titchmarsh, Theory of functions, Oxford, 1932, pp. 218–220] to the effect that if ζ is a regular point and $a_n\to 0$ then $f_n(\zeta)\to f(\zeta)$. In the present paper the author extends these results to points ζ for which $\phi(\zeta)=\lim_{z\to\zeta,|z|<1} f(z)$ exists. The following theorem is typical of those proven. Suppose (a) there exists three real numbers $0<\alpha\leqq 1$, $\rho>0$, and $K>0$ such that $|f'(z)|<K|z-\zeta|^{\alpha-1}$ for $|z|<1$, $|z-\zeta|<\rho$ and (b) $|a_n|<Cn^{-\gamma}$ ($\gamma>0$) for all sufficiently large n (C independent of n). Then for sufficiently large n

$$|f_n(\zeta)-\phi(\zeta)|<\begin{cases}\dfrac{c_1}{\alpha}\,n^{-\alpha\gamma/(2+\alpha)}, & \text{when } 0<\gamma\leqq 1+\tfrac{1}{2}\alpha,\\[2ex] \dfrac{C}{\gamma-1}\,n^{-(\gamma-1)}, & \text{when } 1+\tfrac{1}{2}\alpha\leqq\gamma.\end{cases}$$

Here c_1 depends on C but not on n, α or γ. *V. F. Cowling.*

Kulik, Stephen. A method for approximating the zeros of analytic functions. Duke Math. J. **24** (1957), 137–141.

Let $f(z)$ be analytic in and on a circle C. Let the zeros $a_1, a_2, \cdots, a_p$ of $f(z)$ in the multiplicities $m_1, m_2, \cdots, m_p$ lie in C. Let z be any number closer, say, to a_1 than to any other a_j. Let $g(z)$ be any second function analytic in and on C that has the same a_j as zeros of multiplicity p_j where $0\leqq p_j<m_j$. Let Q_n be defined by the relation

$$(n-1)!Q_n=(-1)^{n-1}[f(z)]^n(d/dz)^{n-1}[g(z)/f(z)].$$

Then $a_1=\lim F_n(z)$, as $n\to\infty$, for $F_n(z)=z-f(z)S_n$, where $S_n=Q_{n-1}/Q_n$ or $S_n=Q_n{}^{-1/n}$. Among the choices of $g(z)$ suggested is $g(z)=f'(z)$, for which z has a nth order convergence to a_1. Recursive formulas are given for the Q_n corresponding to this and other choices of $g(z)$.

M. Marden (Milwaukee, Wis.).

Lense, Josef. Über algebroide und algebromorphe Funktionen. Bayer. Akad. Wiss. Math.-Nat. Kl. S.-B. **1956**, 237–242 (1957).

Vernić's [Diskussion der Sundmanschen Lösung des Dreikörperproblems, Jugoslav. Akad. Znan. Umjet., Zagreb, 1954; MR **16**, 867] definition (loc. cit., p. 102) of the fundamental algebromorphic function is not precise. The author therefore suggests that the formula (44)

(loc. cit., p. 105) be taken as a definition for the fundamental algebromorphic function. The convergence of the infinite product appearing in formula (44) can then be deduced easily from Weierstrass' product formula for integral functions.

Furthermore the author shows that the algebroids constitute a particular case of the algebromorphic functions, namely, the finite-valued ones. The fundamental algebromorphic function, on the other hand, is a particular case of an algebroid, namely, a binomial algebroid.

Vernić's theorem 18 (loc. cit., p. 102), according to which the general solution of the three-body problem is an algebromorphic function of the time t, contradicts an earlier result of Siegel [Ann. of Math. (2) **42** (1941), 127–168; MR **2**, 263], according to which series expansions with irrational exponents arise in a neighborhood of an instant of a triple collision, i.e., the critical point is an isolated but essential (logarithmic) singularity.

E. Leimanis (Vancouver, B.C.).

Siryk, G. V. On conformal mapping of nearby regions. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 5(71), 57–60. (Russian)

Methods due to Warschawski [Quart. Appl. Math. **3** (1945), 12–28; MR **6**, 207] and Laurent'ev [Conformal representations with applications to some problems of mechanics, Gostehizdat, Moscow-Leningrad, 1946] for determining functions which map nearly circular domains conformally on the unit circle are extended to the cases when the domains are "nearly" halfplanes, strips, and circular rings. *W. Seidel.*

Bernardi, S. D. The centroid of analytic mappings. Amer. Math. Monthly **64** (1957), 259–261.

Let the schlicht function $f(z)=a_1z+a_2z^2+\cdots$ map $|z|<r$ conformally onto a region whose centroid is denoted by $C(r)$. It is shown that the locus of the points $C(r)$ corresponding to an interval $0\leq r<R$ of values of r is a straight line if and only if the coefficients $a_1, a_2, \cdots$ are (essentially) real. *P. R. Garabedian.*

Beurling, A.; and Ahlfors, L. The boundary correspondence under quasiconformal mappings. Acta Math. **96** (1956), 125–142.

Let $w=f(z)$ be a quasiconformal mapping of the unit disc $|z|<1$ onto the unit disc $|w|<1$. It was already pointed out by Grötzsch that such a mapping admits an extension to a homeomorphic mapping of the closed discs. A fundamental question arising from this is whether this extension restricted to $|z|=1$ is absolutely continuous (= measurable) in the one-dimensional sense. Some years ago Pfluger [C. R. Acad. Sci. **226** (1948), 623–625; MR **9**, 421] published a note purporting to answer the question in the affirmative but the argument is defective. The authors now answer the question in the negative through the medium of obtaining a characterization of such boundary correspondences. They treat the problem in the following equivalent normalization. They regard quasiconformal mappings of the upper half-plane $y>0$ onto the upper half-plane $v>0$ under which the boundary points at infinity correspond, and (strictly) monotone increasing continuous functions $\mu(x)$ carrying the real axis into itself by the mapping $x\to\mu(x)$, and ask when the latter is a boundary correspondence associated with one of the former quasiconformal mappings. The authors show that a necessary and sufficient condition is that the function μ should satisfy a ρ-condition:

$$\rho^{-1}\leq[\mu(x+t)-\mu(x)][\mu(x)-\mu(x-t)]^{-1}\leq\rho$$

for some constant $\rho\geq 1$ and all real x and t. It is shown that the ρ-condition is essentially a compactness condition. The necessity proof amounts to a study of the effect of the mapping upon the module of quadrangles obtained from the upper half-plane by assigning the vertices $x-t, x, x+t, \infty$, and is quite straightforward. The essential part of the work lies in the sufficiency proof, where there is constructed explicitly a mapping on the upper half-plane $y>0$ having the given boundary values, which is then shown to be quasiconformal. It seems somewhat surprising that the authors have defined quasiconformal mappings only in the differentiable case, since all considerations seem to apply immediately to prove the results also with the more general geometric definition. Finally, the authors give an example to show that a function $\mu(x)$ as above may satisfy a ρ-condition without being absolutely continuous. This shows that the boundary correspondence for quasiconformal mappings need not be measurable. The authors do not go into applications of this result, but it is worth while to point out here several of them which follow from earlier results: (i) the property of a Riemann surface that there exist on it no non-constant bounded analytic functions, is not invariant under quasiconformal mapping [Ahlfors and Beurling, Acta Math. **83** (1950), 101–129; MR **12**, 171]; (ii) the analogue of Fatou's theorem does not hold for bounded quasiconformal mappings [Jenkins, J. Rational Mech. Anal. **5** (1956), 343–352; MR **18**, 26]; (iii) the analogue of the theorem of F. and M. Riesz does not hold for bounded quasiconformal mappings [ibid.]. *J. A. Jenkins.*

Pesin, I. N. Metric properties of Q-quasiconformal mappings. Mat. Sb. N.S. **40(82)** (1956), 281–294. (Russian)

Generalized Q-quasiconformal mappings (Q-qm's) were introduced in Dokl. Akad. Nauk SSSR (N.S.) **102** (1955), 223–224, 865–866 [MR **17**, 26]. The first section of the present paper centers about the theorem that every Q-qm is absolutely continuous. It includes, among others, the proposition that the Q-qm's form a group; that the image of a measurable set, under a Q-qm, is again measurable; that the characteristic function $q(z)$ of a Q-qm is measurable. The paper also deals with nullsets, removable sets, and sets of uniqueness. Examples: If Ω is a domain and E is a set of class N_{SB} (notation of analytic function theory) contained in Ω, then every Q-qm of Ω carries E onto a set of class N_{SB}; if a set E lies on the Q-q image of a line segment, it belongs either to both or to neither of the two classes N_D and N_{SD}. *G. Piranian.*

Yûjôbô, Zuiman. Supplements and corrections to my paper: On absolutely continuous functions of two or more variables in the Tonelli sense and quasi-conformal mappings in the A. Mori sense. Comment. Math. Univ. St. Paul. **5** (1956), 33–36.

The last two theorems of an earlier paper by the same author [same Comment. **4** (1955), 67–92; MR **17**, 836] are stated without proof in a slightly modified form.

K. Strebel (Freiburg).

Oikawa, Kôtaro. On the prolongation of an open Riemann surface of finite genus. Kôdai Math. Sem. Rep. **9** (1957), 34–41.

Let F be a non-compact Riemann surface of finite

genus $g \geq 2$. Let $\mathfrak{P}(F)$ denote the set of pairs (W, f), where W is a compact Riemann surface of genus g and f is a univalent conformal map of F into W. On fixing a pair (W_0, f_0) it is possible to associate with W_0 a Teichmüller space [cf. Ahlfors, J. Analyse Math. 3 (1954), 1–58, 207–208; MR 16, 348] and a map of $\mathfrak{P}(F)$ into the Teichmüller space, $(W, f) \to \langle W, \alpha_f \rangle$, where $\langle W, \alpha_f \rangle$ is the equivalence class containing (W, α_f), α_f being the homotopy class of orientation preserving transformations of W_0 onto W induced by $f \circ f_0^{-1}$. It is shown that the image of $\mathfrak{P}(F)$ with respect to the map so defined is a compact connected subset of the Teichmüller space. Essential use is made of the theory of quasi-conformal maps. *M. Heins.*

Sörensen, Werner; et Bader, Roger. Noyaux de Green-de Rham sur les surfaces de Riemann. C. R. Acad. Sci. Paris **244** (1957), 1309–1311.

Explicit formulas are given in terms of abelian differentials for Green-de Rham kernels for a compact Riemann surface and for a relatively compact region with regular frontier of a non-compact Riemann surface. The role played by such formulas for the Cousin problem for a Riemann surface is indicated. *M. Heins.*

Ozawa, Mitsuru. On Riemann surfaces admitting an infinite cyclic conformal transformation group. Kôdai Math. Sem. Rep. **8** (1956), 152–157.

Let W denote a non-compact Riemann surface admitting an infinite cyclic group $\mathfrak{G}$ of conformal automorphisms. Let W have precisely two distinct ideal boundary elements and be such that $T^n p$ and $T^{-n} p$, where $p \in W$ and T is a generator of $\mathfrak{G}$, tend respectively to the two ideal boundary elements as $n \to +\infty$. It is proposed to determine a necessary and sufficient condition that W admit a meromorphic function of finite valence. The author establishes a necessary and sufficient condition which is expressed in terms of the Riemann matrix associated with a canonical basis of the one-dimensional homology group of the compact Riemann surface W mod $\mathfrak{G}$. *M. Heins* (Providence, R.I.).

Clunie, J. On a theorem of Noble. J. London Math. Soc. **32** (1957), 138–144.

Let $f(z)$ be an entire function of genus 0 and lower order σ, $0 < \sigma < 1$, with all but a finite number of its zeros a_n in the upper half plane. Let $\Re(a_n) = o(|a_n|)$ and $\log |f(x)| \sim (\frac{1}{2}\pi\lambda \csc \frac{1}{2}\pi\rho) x^\rho$. Then $n(r) \sim \lambda r^\rho$. This implies Noble's theorem [Proc. Cambridge Philos. Soc. **47** (1951), 22–37; MR **12**, 600] in which lower order is replaced by order and the condition on the location of the zeros is omitted. An example shows that $\sigma = 1$ is inadmissible. *R. P. Boas, Jr* (Evanston, Ill.).

Arsove, Maynard G. Proper bases and automorphisms in the space of entire functions. Proc. Amer. Math. Soc. **8** (1957), 264–271.

The linear space Γ of all entire functions, topologized by the metric of uniform convergence on compact sets, has been investigated by V. Ganapathy Iyer [see MR **10**, 380; **12**, 108; **14**, 657; **17**, 1225]. A basis in Γ is a sequence $\{\alpha_n(z)\}$ of entire functions such that every entire function $f(z)$ can be represented as $f(z) = \sum_0^\infty c_n \alpha_n(z)$ (c_n are complex numbers; convergence according to the topology on Γ). The best known basis being $\{\delta_n(z)\} \equiv \{z^n\}$ (power series), Ganapathy Iyer introduced the notion "proper basis" for bases satisfying a characteristic property of $\{\delta_n\}$. The author proposes a modified definition of "proper bases", making them more closely similar in their behaviour to $\{\delta_n\}$. The investigation of these bases is facilitated by the use of $M_n(R) = \max_{|z|=R} |\alpha_n(z)|$ instead of $|\alpha_n; R| = \sum_k |A_{nk}| R^k$ (where $\alpha_n(z) = \sum_k A_{nk} z^k$), and the inequalities $M_n(R) \leq |\alpha_n; R| \leq 2M_n(2R)$. The linearly homeomorphic images of Γ in itself are then characterized as the closed subspaces admitting proper bases. Furthermore, the proper bases in Γ are the sequences which map into $\{\delta_n\}$ under automorphisms of Γ. *B. A. Amirà* (Jerusalem).

Mikhail, M. N. The behaviour of the random meromorphic function at its zeros. I, II. Nederl. Akad. Wetensch. Proc. Ser. A. **60** = Indag. Math. **19** (1957), 88–95, 96–103.

J. E. Littlewood and A. C. Offord [Ann. of Math. (2) **49** (1948) 885–952; **50** (1947), 990–991; MR **10**, 692] discussed the subject of "pits behavior" of the family F of entire functions defined by $f(z, t) = \sum r_n(t) a_n z^n$, where $\sum a_n z^n$ is a given entire function, of finite non-zero order, and the $r_n(t)$ denote the Rademacher functions, used here to superimpose a "random" factor ± 1. The author investigates the subject of "pits behavior" of a family G of meromorphic functions $g(z, t)$, defined by

$$g(z, t) = \sum r_{2n}(t) b_n z^n / \sum r_{2n+1}(t) c_n z^n,$$

where $\sum b_n z^n$ and $\sum c_n z^n$ are two given entire functions of finite non-zero order and the $r_n(t)$ denote the functions of Rademacher. The author shows that the general conclusions of Littlewood and Offord remain true for meromorphic functions of the class G: almost all functions of G display the "pits behavior", that is, are exponentially large except in pits of exponentially small areas. Some of these pits may be shallow. Following Littlewood and Offord, it is shown that, if D is any bounded set of a-values containing $a = 0$, then "almost all" g of G have, in all but a finite number of pits, as many a-values as zeros. *A. Edrei* (Syracuse, N.Y.).

Batyrev, A. V. Approximate solution of the problem of Riemann-Privalov. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 5(71), 71–76. (Russian)

Two methods for the approximate solution of Riemann's boundary value problem are given. [This problem has been discussed previously and an exact solution has been given by F. B. Gakhov, Dokl. Akad. Nauk SSSR (N.S.) **67** (1949), 601–604; **80** (1951), 705–708; MR **11**, 169; **13**, 545.] The first method consists in replacing the boundary conditions $\phi^+(t) = G(t)\phi^-(t)$ on the closed rectifiable curve L, where $G(t)$ is continuous on L, ϕ^+ is the required solution in the interior of L and ϕ^- the one in the exterior, by the following approximate condition: $\phi_n^+(t) = t^{-n} P_n^+(t) P_n^-(t) \phi_n^-(t)$, where P_n^+ and P_n^- are polynomials of degree n (n must be sufficiently large) whose roots lie entirely in the interior or exterior of L respectively. Using analytic continuation and Liouville's theorem the solution is then

$$\phi_n^+(z) = C P_n^-(z), \quad \phi_n^-(z) = C z^n / P_n^+(z) \quad (C = \text{const.}).$$

In the second method one uses the boundary condition in logarithmic form and approximates this relation by

$$\log \phi_n^+(t) - \log \phi_n^-(t) = p_n(t) + q_n(t^{-1}),$$

where p_n and q_n are polynomials of degree n. In this case the approximate solution has the form

$$\phi_n^+(z) = C \exp(p_n(z)), \quad \phi_n^-(z) = C \exp(-q_n(z^{-1})).$$

It is then shown that for both methods $\phi_n^+(z)$ and $\phi_n^-(z)$

converge uniformly in the corresponding closed regions to the exact solution. The author indicates also methods for finding the polynomials $P_n^+(t)$, $P_n^-(t)$, $p_n(t)$, $q_n(t^{-1})$ by using finite segments of the Laurent series or expansions into Faber polynomials. *U. W. Hochstrasser.*

Goodman, A. W. Univalent functions and nonanalytic curves. Proc. Amer. Math. Soc. **8** (1957), 598–601.

The author uses known conditions that a univalent function be starlike in order to construct in parametric form a convex closed curve containing no analytic arc. *P. R. Garabedian* (London).

Halim, E. On the order of simple sets of polynomials and associated sets. Proc. Math. Phys. Soc. Egypt **5** (1955), no. 3, 55–61 (1957).

Let $\{p_n(z)\}$ be a simple set of polynomials of order ω, $p_n(z)=\sum_{k=0}^{n} p_{nk}z^k$ ($p_{nn}\neq 0$). The author proves that if

$$\liminf_{n\to\infty} \frac{\log |p_{nn}|}{n\log n}=a,\quad \limsup_{n\to\infty} \frac{\log |p_{nn}|}{n\log n}=b,$$

and if the reciprocal set $\{\bar{p}_n(z)\}$ is of order $\bar{\omega}$, then

$$\min\{\tfrac{1}{2}(\omega+a),\ \tfrac{1}{2}(\omega+a-b)\}\leqq\bar{\omega}\leqq\max\{2\omega-a+b,\ 2\omega+b\}.$$

In addition, if a simple set $\{p_n(z)\}$ of order ω is the product set of the simple sets $\{p_n^{(1)}(z)\}$ and $\{p_n^{(2)}(z)\}$ of orders ω_1, ω_2, respectively, and if

$$\liminf_{n\to\infty} \frac{|p_{nn}^{(2)}|}{n\log n}=a,\quad \limsup_{n\to\infty} \frac{|p_{nn}^{(2)}|}{n\log n}=b,$$

then $\omega\leqq\max\{\omega_1+\omega_2-a,\ \omega_1+2\omega_2-a+b\}$.

M. Newman (Washington, D.C.).

Arens, Richard. Cauchy integral for functions of several variables. Tôhoku Math. J. (2) **8** (1956), 268–272.

Ist P ein abgeschlossenes analytisches Polyeder, das relativ-kompakt in einem Gebiet G des n-dimensionalen komplexen Zahlen-Raumes C^n liegt, so lässt sich jede in P holomorphe Funktion im Innern von P durch das sog. Weilsche Integral repräsentieren. Die Weilsche Integraldarstellung hat ähnliche Eigenschaften wie die Cauchysche Integraldarstellung der holomorphen Funktionen einer Veränderlichen, doch ist beim Weilschen Integral nur über einen Teil des Randes von P, die sog. ausgezeichnete (reell n-dimensionale) Randfläche B zu integrieren. Dabei treten Schwierigkeiten auf, die durch Singularitäten von B bedingt werden. Man hat auf verschiedene Weise versucht, diese Schwierigkeiten zu beheben. U.a. hat M. Hervé im Séminaire H. Cartan Ecole Norm. Sup., 1951/52, Exposé VI [MR **16**, 233] gezeigt, dass man anstatt über B auch über ein volldimensionales Gebiet in der Nähe von B integrieren darf. Der Verf. der vorl. Arbeit beweist nun, dass man immer in beliebiger Nähe von B eine singularitätenfreie analytische n-dimensionale Kette C finden kann und dass man im Weilschen Integral die Integration über B durch eine Integration über C ersetzen darf. *H. Grauert* (Princeton, N.J.).

Hua, Loo-Keng. On the Riemannian curvature in the space of several complex variables. Schr. Forschungsinst. Math. **1** (1957), 245–263.

Let D be a domain in the space of n complex variables $z^1, \cdots, z^n$. Let $\varphi_0, \varphi_1, \cdots$ be a sequence of analytic functions on D which have no common zero and such that $\{\varphi_j/\varphi_0\}$ contains n independent functions; form the Bergmann kernel $K(z,\bar{z})=\sum \varphi_j(z)\overline{\varphi_j(z)}$. (We assume the series converges uniformly on compact sets.) We introduce on D an hermitian metric by means of the differential form $T_q dz^i d\bar{z}^j$, where $T_q=\partial^2 \log K(z,\bar{z})/\partial z^i \partial \bar{z}^j$; let R be the associated Riemannian curvature. Th. A: $R\leq 2$. (Note that the ">" in (1.9) should be "$\leq$".) Th. B: If $K=$ const d, where d is the volume density for T_q, then $R\geqq -n$. ($K=$const d is valid if, e.g., D admits a transitive group of analytic transformations.) Th. C: Let D be bounded, and let $\{\varphi_j\}$ be a complete orthornomal system for the analytic functions of $L_2(D;\ dx^1dy^1\cdots dx^ndy^n)$. Suppose for each boundary point z of D there is an analytic $f\in L_2(D;\ dx^1\cdots dy^n)$ which is infinite at z. If R is constant, D is analytically homeomorphic to the unit ball $\sum |z_j|^2<1$. — All the proofs are explicit calculations. A special case of Th. A was found by Fuchs [Mat. Sb. N.S. **2(44)** (1937), 567–594]. {The author states Th. C for non-continuable domains, but the proof is only valid for domains non-continuable in the L_2 sense described above. In fact, the example $n=1$, $D=$punctured unit ball shows that the hypothesis "non-continuable" is not sufficient.}

L. Ehrenpreis (Princeton, N.J.).

Gundlach, Karl-Bernhard. Modulfunktionen zur Hilbertschen Modulgruppe und ihre Darstellung als Quotienten ganzer Modulformen. Arch. Math. **7** (1956), 333–338.

By using a recent result of Siegel [Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. IIa. **1955**, 71–77; MR **17**, 530], the author gives a short proof of the following theorem: Every set of $n+1$ modular functions belonging to the Hilbert modular group on n variables is algebraically dependent. Every modular function is the quotient of two entire modular forms. *J. Lehner.*

Danilyuk, I. I. On quasianalytic functions of many variables on manifolds of an even number of dimensions. Dokl. Akad. Nauk SSSR (N.S.) **109** (1956), 434–437. (Russian)

A complex function $f=u^1+iu^2$ of many complex variables is called quasianalytic if it satisfies an elliptic system $\partial u^1/\partial x^j=a_j\partial u^2/\partial x^1+b_j\partial u^2/\partial x^2$ ($j=1, 2$) for each complex variable $z=x^1+ix^2$, and is continuous, except at some isolated points, in all the variables. The author constructs a Hilbert space of quasianalytic functions, exhibits formulas and draws some conclusions based on the theory of continuous groups. No proofs are given.

D. C. Kleinecke (Albuquerque, N.M.).

See also: Linear Algebra: Parodi. **Geometry of Numbers:** Frank. **Harmonic Functions, Complex Functions:** Ivanov; Rauch. **Special Functions:** Myrberg; Maass. **Approximations, Orthogonal Functions:** Vertgeim; Moser and Wyman; Davis. **Ordinary Differential Equations:** Röhrl. **Partial Differential Equations:** Séminaire Schwartz; Kreyszig. **Difference Equations, Functional Equations:** Alander; Osserman. **Banach Spaces, Banach Algebras, Hilbert Spaces:** Schieferdecker; Sz.-Nagy and Korány. **Complex Manifolds:** Dalla Volta; Griesel; Holmann; Grauert and Remmert. **Numerical Methods:** Barrett.

Geometric Analysis

See: Measure and Integration: Cesari. **Differential Geometry:** Whitney; Gugenheim and Spencer. **Manifolds, Connections:** Slebodziński; Sasayama. **Complex Manifolds:** Jongmans; Hirzebruch. **Quantum Mechanics:** Winogradzki.

Harmonic Functions, Convex Functions

Duffin, R. J. A note on Poisson's integral. Quart. Appl. Math. **15** (1957), 109–111.

For a continuous function f given on the boundary of a convex domain, a continuous extension of f into the interior is constructed in two steps: (1) define the linear function g on each line segment with the correct values on the boundary points; (2) at each interior point P form the average of all $g(P)$. The resulting function $F(P)$ is a sum of two integrals F_1+F_2, where F_1 is a harmonic function. When the domain is a circle or sphere, F_2 reduces to a constant. For nearly circular domains F is an approximate solution of the Dirichlet problem.

Yu Why Chen (Detroit, Mich.).

Brelot, Marcel; et Choquet, Gustave. Le théorème de convergence en théorie du potentiel. J. Madras Univ. Sect. B. **27** (1957), 277–286.

Les auteurs donnent une nouvelle démonstration du théorème de Cartan-Brelot (viz: étant donnée une famille de mesures $\mu_i \geqq 0$, il existe une mesure $\geqq 0$ dont le potentiel est égal, sauf sur un ensemble de capacité extérieure nulle, à la borne inférieure des potentiels des μ_i). Cette démonstration n'utilise pas la notion d'énergie et vaut pour des "noyaux" assez généraux. On s'y sert notamment du lemme suivant: soit E un espace topologique à base dénombrable d'ouverts, et soit $(f_i)_{i\in I}$ une famille de fonctions numériques semi-continues inférieurement (à valeurs éventuellement infinies), définies dans E; alors il existe une partie dénombrable $J \subset I$ jouissant de la propriété suivante: si g est semi-continue inférieurement et $g \leqq f_i$ pour $i \in J$, alors $g \leqq f_i$ pour tout $i \in I$.

Les auteurs démontrent le théorème suivant. Soit E un espace localement compact à base dénombrable; soit $G(x, y)$ un "noyau" semi-continu inférieurement sur $E \times E$, à valeurs $\geqq 0$ éventuellement infinies. Définissons le potentiel $G_\mu(x) = \int G(x, y)d\mu(y)$ pour toute mesure $\mu \geqq 0$. Supposons que le noyau $G^*(x, y) = G(y, x)$ soit "régulier", i.e. pour toute mesure $\nu \geqq 0$ à support compact telle que la restriction de G_ν^* au support de ν soit continue finie, G_ν^* est continu fini dans E. Soit $(\mu_i)_{i\in I}$ une famille de mesures $\geqq 0$ telles que: (a) les potentiels G_{μ_i} forment une famille filtrante décroissante; (b) pour tout compact K, $\sup_i \int_K d\mu_i < +\infty$; (c) pour toute mesure $\nu \geqq 0$ à support compact telle que G^* soit fini et continu, et pour tout $\varepsilon > 0$, il existe un compact $K \subset E$ tel que $\int_{E-K} G_\nu^* d\mu_i \leqq \varepsilon$ pour tout $i \in I$. Alors il existe une suite d'indices $i_n \in I$ et une mesure $\mu \geqq 0$ jouissant des propriétés suivantes: (A) $\lim \mu_{i_n} = \mu$ au sens de la topologie vague; (B) la suite des potentiels G_{λ_n} (où λ_n désigne μ_{i_n}) est décroissante et converge vers G_μ sauf sur un ensemble exceptional de G^*-capacité intérieure nulle; (C) $G_\mu \geqq G_{\mu_i}$ pour tout $i \in I$ et en tout point de E. Si de plus les ensembles boréliens sont G^*-capacitables au sens de Choquet, alors l'ensemble exceptionnel est de capacité extérieure nulle.

H. Cartan (Paris).

Ivanov, V. K. Distribution of singularities of a potential and a three-dimensional analogue of Polya's theorem. Mat. Sb. N.S. **40(82)** (1956), 319–338. (Russian)

In this article the following problem is considered. Given a finite mass distributed in a finite region D lying in a half space $z>0$. Find the distance H from the plane $z=0$ to the set of singularities of the harmonic function obtained by analytic continuation of the exterior potential V of this mass distribution into the interior of D, if the derivative V_z is given on the plane $z=0$.

The result is expressed by the formula:

$$H = -\sup_{\psi} \lim_{r\to\infty} \frac{\log v_1(r\cos\psi, r\sin\psi)}{r},$$

where

$$v_1(\xi, \eta) = (2\pi)^{-1} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(x\xi+y\eta)} V_z(x, y, 0)dxdy;$$

an analogous result is given for the two dimensional case. The proof of this relation makes use of the theory of quaternion functions. In its course a three-dimensional analogue to a theorem by Polya [Math. Z. **29** (1929), 549–640] is derived, which says the following: Let V be the external potential of a finite mass distribution $M(x, y, z)$ filling a finite region D; denote by $K(\theta, \varphi)$ the supporting function of the convex hull of the singularities of the analytic continuation of V into D; and let $h(\psi, \theta, \varphi)$ be the indicator function of the characteristic function $F(\mathbf{p})$ of the mass distribution $M(x, y, z)$ ($F(\mathbf{p}) = \int_D \exp(\mathbf{pr})dM(x, y, z)$). Then the relation $K(\theta, \varphi) = \sup_\psi h(\psi, \theta, \varphi + \frac{1}{2}\pi)$ holds. *U. W. Hochstrasser.*

Rauch, H. E. Harmonic and analytic functions of several variables and the maximal theorem of Hardy and Littlewood. Canad. J. Math. **8** (1956), 171–183.

This paper presents applications to harmonic and analytic functions of an extension of the maximal theorem of Hardy and Littlewood [Acta Math. **54** (1930), 81–116], that were essentially obtained in the author's doctoral dissertation and were described earlier [C. R. Acad. Sci. Paris **227** (1948), 887–889; MR **10**, 309]. The results and methods are closely related to those obtained independently in the paper reviewed below. The basic tool is a covering lemma for subsets of a metric space with a measure, from which the extension of the maximal theorem is obtained. This extension is then applied to obtain theorems on radial suprema of analytic and harmonic functions on spheres and polycylinders. The main results are the following. Let $f(z)$, $z=(z_1, \cdots, z_n)$, be analytic in $|z| = [\sum |z_i|^2]^{\frac{1}{2}} < 1$ and such that the integral of $|f(rz)|^\lambda$, $\lambda > 0$, on $|z| = 1$ is bounded by M for $0 < r < 1$; then $\sup_r |f(rz)|^\lambda$ is integrable on $|z| = 1$ and its integral does not exceed αM, where α depends only on λ and n. A similar result holds if the z_i are real, f is harmonic and $\lambda > 1$. Let $f(z)$ be analytic in the polycylinder $|z_i| \leqq 1$ and let the integral of $|f(rz)|^\lambda$, $\lambda > 0$, on $|z_i| = 1$ be bounded by M in $r < 1$. Then $\sup |f(r_1 z_1, \cdots, r_n z_n)|^\lambda$, $|z_i| = 1$, $(1-r_i)(1-r_j)^{-1} \leqq K$, is integrable on $|z_i| = 1$ and its integral does not exceed αM, where α depends only on λ, n and K. A similar result holds if f is polyharmonic and $\lambda > 1$.

A. P. Calderón (Cambridge, Mass.).

Smith, K. T. A generalization of an inequality of Hardy and Littlewood. Canad. J. Math. **8** (1956), 157–170.

The present paper is concerned with an extension of the maximal theorem of Hardy and Littlewood [see also the preceding review]. The generalization depends on a covering lemma due to N. Aronsjazn and the author [Ann. Inst. Fourier, Grenoble **6** (1955–1956), 125–185; MR **18**, 319], analogous to others used by Wiener [Duke Math. J. **5** (1939), 1–18], Rauch [see the paper reviewed above] and the reviewer [Ann. of Math. (2) **58** (1953), 182–191; MR **14**, 1071]. The main result can be stated as follows. Let B be a metric space with a Borel measure such that, if $S(x, r)$ denotes a sphere with center at x and radius

r and $|S(x, r)|$ is its measure, then $|S(x, 4r)| \leqq K|S(x, r)|$, $|S| \to 0$ implies $r \to 0$ and, $r \to \infty$ implies $|S| \to \infty$ uniformly in x. Then if $\bar{f}(x) = \sup_r |S|^{-1} \int_{S(x,r)} f(y)dy$, $f \in L^p$ implies $\bar{f} \in L^p$, $1 < p < \infty$, and $f \in L \log^+ L$ implies $\bar{f} \in L$ locally, and corresponding inequalities hold for the norms of f and $\bar{f}$. This theorem is then applied to obtain results on boundary values of harmonic and subharmonic functions in domains with smooth boundaries and strong differentiability of indefinite integrals on products of metric spaces with measures as above. In the case the domain is a sphere some of the results can be briefly described as follows. Let u be a function harmonic in the interior of the sphere, $\tilde{u}(x)$ the supremum of $|u|$ on the radius through the point x of the surface of the sphere, and I_p the least upper bound of the integral of $|u|^p$ on concentric spherical surfaces. Then for $1 < p < \infty$, $I_p < \infty$ implies $\tilde{u} \in L^p$. If $p = 1$ then $\tilde{u} \in L^{1-\varepsilon}$ for every $\varepsilon > 0$. Similar results hold if u is subharmonic and non-negative. The case of a subharmonic not necessarily non-negative u is also discussed.

A. P. Calderón (Cambridge, Mass.).

Magenes, Enrico. Sulla teoria del potenziale. Rend. Sem. Mat. Univ. Padova **24** (1955), 510–522.

The author considers Newtonian potentials of simple and double layers distributed over a surface F of class C_2 and discusses the existence of limits on F of the double layer potential and of the first order derivatives of the simple layer potential. On the assumption of mere integrability with respect to the surface area of the layer density, the author proves the equivalence almost everywhere of the existence of limits and the existence of certain principal value integrals over F. For the proof of existence of the principal value integral in the case of a simple layer the reader is referred to Ann. Mat. Pura Appl. (4) **40** (1955), 143–160 [MR **17**, 1090]. In the case of a double layer, the proof appears faulty to the reviewer. It depends on two continuous linear mappings from a Banach space B_1 into another B_2 having the same range, and this is not implied in general, as the author seems to assert, by the fact that the ranges of the transformations coincide when these are restricted to a dense subspace of B_1. Finally, the author indicates how to extend his results to the case of similar potentials with respect to fundamental solutions of elliptic second order partial differential equations. *A. P. Calderón* (Cambridge, Mass.).

Kawakami, Yoshiro. Theorems on subharmonic functions in the unit circle. Kōdai Math. Sem. Rep. **8** (1956), 158–163.

Suppose that $u(z)$ is subharmonic in $|z| < 1$ and that

$$\int_0^{2\pi} |u(re^{i\theta})| d\theta = O(1)$$

as $r \to 1$. Littlewood [Proc. London Math. Soc. (2) **28** (1928), 383–394] proved that the radial limit $u(e^{i\theta}) = \lim_{r \to 1} u(re^{i\theta})$ exists on a set E of θ in $[0, 2\pi]$, having measure 2π. If in addition

$$\frac{d}{dr} \int_0^{2\pi} u(re^{i\theta}) d\theta = O(1-r)^{-\lambda},$$

as $r \to 1$, where $0 < \lambda < 1$, then Tsuji [Comment. Math. Univ. St. Paul, **5** (1956), 3–16; MR **18**, 122] showed that such a set E can be chosen so that for each θ on E we have (*) $u(e^{i\theta} - te^{i(\theta-\psi)}) \to u(e^{i\theta})$ as $t \to 0$ for almost all ψ in $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$. The present author now shows that the limit in (*) exists uniformly in $-\frac{1}{2}\pi + \delta < \psi < \frac{1}{2}\pi - \delta$ for any $\delta > 0$, as $t \to 0$ outside a set of finite logarithmic length.

W. K. Hayman (London).

Górski, J. On a certain sequence which converges to the generalized transfinite diameter of a plane set. Prace Mat. **2** (1956), 152–158. (Polish. Russian and English summaries)

The paper solves a problem of Leja. Let E be a bounded and closed plane set and $f(z)$ a real continuous function defined on E. Let $\{\zeta_0, \zeta_1, \cdots, \zeta_n\} = \zeta$ be a set of $n+1$ different points of E. One forms the expression

$$U(\zeta_0, \zeta_1, \cdots, \zeta_n) = \prod_{0 \leq j < k \leq n} |\zeta_j - \zeta_k| / \exp(f(\zeta_j) + f(\zeta_k))$$

and denotes by $\eta_0, \eta_1, \cdots, \eta_n$ a set of $n+1$ points of E such that $U(\eta_0, \eta_1, \cdots, \eta_n) = \sup_{\zeta \subset E} U(\zeta_0, \zeta_1, \cdots, \zeta_n)$. The limit

$$\lim_{n \to \infty} (U(\eta_0, \eta_1, \cdots, \eta_n))^{1/s} = d(E) \geqq 0, \text{ where } s = \binom{n+1}{2},$$

exists; it is called the generalized transfinite diameter of E. Consider the product

$$\Delta_n^{(j)} = \prod_{k=0, k \neq j}^{n} |\eta_j - \eta_k| / \exp(f(\eta_j) + f(\eta_k)) \quad (j = 0, 1, \cdots, n).$$

Let $\Delta_n^{(0)} \leqq \Delta_n^{(j)}$ $(j = 0, 1, \cdots, n)$. The limit $\lim_{n \to \infty} (\Delta_n^{(0)})^{1/s}$ exists and is equal to $d(E)$ if (1) E is a boundary of a domain which contains the point $z = \infty$ in its interior and (2) E is a sum of continua. (From the author's summary.)

S. M. Ulam.

Walter, Wolfgang. Verallgemeinerte Laplace-Operatoren und Potentiale. Math. Z. **67** (1957), 38–48.

Let μ be a positive, countably additive function on the Borel sets of the n-dimensional space, which vanishes outside of a bounded set, and let for an integer p, $\gamma_p(r) = C_{np} r^{2p-n}$ unless $n = 2, 4, \cdots, 2p$, in which case $\gamma_p(r) = C_{np} r^{2p-n} \log r$; C_{np} is a proper numerical factor. Then $F(x) = \int \gamma_p(|y-x|) d\mu(y)$ is called the "p-potential" of the mass distribution μ. The author proves that at each point x where μ has the density $D\mu(x)$, one has the relation $\Delta_*^p F(x) = -D\mu(x)$. Here $\Delta_*^p F(x)$ stands for one of the generalized Laplace operators of order p (which for smooth F reduce to the pth iterate of the ordinary Laplace operator) described by many authors, and in particular those given by the author himself [Jber. Deutsch. Math. Verein. **59** (1957), Abt. 1, 93–131; MR **19**, 283].

G. G. Lorentz (Detroit, Mich.).

Ninomiya, Nobuyuki. Sur le principe de continuité dans la théorie du potentiel. J. Inst. Polytech. Osaka City Univ. Ser. A. **8** (1957), 51–56.

In an m-dimensional Euclidean space R^m the potential of a measure μ of R^m is defined with respect to the kernel K by $U^\mu(x) = \int K(x-y) d\mu(y)$, where x and y refer to the points with the coordinate $(x_1, x_2, \cdots, x_m)$ and $(y_1, y_2, \cdots, y_m)$, respectively. A study is made of kernels satisfying the continuity principle. It is proved that kernels satisfying certain stated conditions will satisfy the continuity principle. *J. E. Rosenthal.*

See also: Integral Transforms: du Plessis. **Partial Differential Equations:** Walter; Campanato. **Fluid Mechanics, Acoustics:** Garabedian. **Programming, Resource Allocation, Games:** Fan, Glicksberg and Hoffman.

Special Functions

Fekete, M.; and Walsh, J. L. On restricted infrapolynomials. J. Analyse Math. **5** (1956/57), 47–76.

Given a pointset S which is bounded finite or infinite closed, and is comprised of at least $n-k$ points ($k<n$). Given also the class K of all polynomials

$$A(z)=z^n+a_1z^{n-1}+\cdots+a_kz^{n-k}+a_{k+1}z^{n-k+1}+\cdots+a_n,$$

where $a_1, a_2, \cdots, a_k$ are given complex numbers, while $a_{k+1}, \cdots, a_n$ are arbitrary constants. A "k-fold restricted infrapolynomial on the set S" is defined as the $A(z)$ which on S has no underpolynomial belonging to K. Such infrapolynomials may be constructed by selecting $a_{k+1}, \cdots, a_n$ so that $A(z_j)=0$ on a set of points z_j ($j=1, 2, \cdots, n-k$) on S. As the authors prove, such an infrapolynomial $A(z)$ cannot have more than k zeros at all of which S subtends an angle less than $\pi/(k+1)$. This theorem follows from the fact that, if $A(z)$ has a factor $B(z)$ non vanishing in S, then in S there are 2 or 3 points z_k such that $\sum \lambda_k B(z_k)=0$ ($\lambda_k>0$, $\sum \lambda_k=1$). Similar considerations lead to the result: a polynomial $D(z)=C(z)+d_{k+1}$, where $C(z)=z^{k+1}+d_1z^k+\cdots+d_kz$, is a k-fold restricted infrapolynomial for a compact pointset S containing at least two points if and only if $w=C(z)$ maps S upon a region whose convex hull contains $w=0$. The paper closes with some theorems about a one-fold restricted infrapolynomial, which is essentially one whose zeros have a prescribed centroid.

M. Marden (Milwaukee, Wis.).

García, Carlos Ibáñez. Study of the polynomials $S_n(x)$ of R. San Juan and of the order of $|D^n e^{-e^x}|^{1/n}$ in the interval $(0, +\infty)$. Rev. Acad. Ci. Madrid **50** (1956), 471–516. (Spanish)

This thesis, dedicated to R. San Juan, is concerned with a detailed discussion of the properties of the polynomials $S_n(x)$ introduced by him [Rev. Mat. Hisp.-Amer. (4) **13** (1953), 128–145; MR **15**, 112]. The following section headings indicate the scope of the paper: I. Definition and primary properties of the $S_n(x)$; II. Integral relations for the $S_n(x)$; III. Orthogonality of the $S_n(x)$ with respect to $e^{-x}x^k$ in $(0, +\infty)$; IV. The non-orthogonality of $S_n(x)$ with respect to any $q(x)$ in (a, b); V. Orthonormalisation of $\{e^{-x}x^kS_n(x)\}$ in $(0, +\infty)$; VI. Expression of $S_n(x)$ by means of the $S_n(2x)$; VII. A relation between $S_n(x)$ and the $L_1^{(\alpha)}(x)$; VIII. Expression of $S_n(x^2)$ in terms of the $H_n(x)$ of Hermite; IX. Bounds for $|D^n e^{-e^x}|^{1/n}$ in $(0, +\infty)$; X. Summary. There are a few misprints and some faulty printing of suffixes.

R. Wilson.

Dickinson, David. On Lommel and Bessel polynomials. Proc. Amer. Math. Soc. **5** (1954), 946–956.

Lommel polynomials are defined as the coefficients $R_{n,\nu}(x)$ in the relations (J_s are Bessel's functions):

$$J_{\nu+n}(x)=J_\nu(x)R_{n,\nu}(x)-J_{\nu-1}(x)R_{n-1,\nu+1}(x)$$

[Watson, G. N. A treatise on the theory of Bessel functions, 2nd ed., Cambridge, 1944; MR **6**, 64]. The author defines modified Lommel polynomials $h_{n,\nu}(x)$ by the equation $h_{n,\nu}(x)=R_{n,\nu}(1/x)$. These polynomials may be represented by

$$h_{n,\nu}(x)=$$

$$(\nu)_n(2x)^n{}_2F_3(-n/2, (-n+1)/2;\ \nu, -n, 1-\nu-n;\ -1/x^2),$$

where $(\nu)_n=\Gamma(\nu+n)/\Gamma(\nu)$.

Suppose $\nu>0$, and let $j_{\nu,n}$ ($n=0, \pm1, \pm2, \cdots$) be the non-zero roots, in order of magnitude, of the function $J_\nu(x)$, where $j_{\nu,1}$ is the smallest positive root; also let $d\alpha_\nu(x)$ be the step function having an increase of $(j_{\nu-1,n})^{-2}$ at the point $(j_{\nu-1,n})^{-1}$. The author proves the following orthogonality relations for the functions $h_{n,\nu}(x)$:

$$\int_a^b h_{n,\nu}(x)h_{m,\nu}(x)d\alpha_\nu(x)=\begin{cases}0 & \text{for } m\neq n,\\ [2^{n+1}(\nu)_{n+1}]^{-1} & \text{for } m=n,\end{cases}$$

where the limits of integration are given by $a=(j_{\nu-1,0})^{-1}$ and $b=(j_{\nu-1,1})^{-1}$. As the author points out, these relations were established independently by H. O. Pollak and G. H. Wannier. The polynomials $h_{n,\nu}(x)$ are connected with Bessel's polynomials $y_n(x)$ in the following way:

$$h_{n,s+\frac{1}{2}}(x)=\tfrac{1}{2}[i^{-n}y_{n+s}(ix)y_{s-1}(-ix)+i^ny_{n+s}(-ix)y_{s-1}(ix)].$$

L. M. Gluskin (RŽ Mat 1956, no. 1464).

Chihara, T. S. On quasi-orthogonal polynomials. Proc. Amer. Math. Soc. **8** (1957), 765–767.

Utilisant une remarque de M. Rosenthal [mêmes Proc. **4** (1953), 600–602; MR **15**, 512] relative à la classe des polynomes

$$R_n{}^\nu(x)=(-1)^n e^{x^2}\frac{d^n}{dx^n}(x^\nu e^{-x^2}),$$

qui satisfont aux relations de quasi orthogonalité

$$(1)\qquad \int_{-\infty}^{+\infty} e^{-x^2}R_m{}^\nu(x)R_n{}^\nu(x)dx=0 \text{ pour } m\neq n\pm 2j,$$

l'auteur propose une extension de la classique formule précédente en définissant à partir d'une suite de polynomes $p_n(x)$ orthonormés quelconques sur (a, b) la nouvelle suite

$$(2)\qquad q_n(x)=\sum_{j=0}^{k} a_{n,j}p_{n-jr}(x)\quad (a_{n,j} \text{ constants})$$

(p_{-m} étant choisi nul pour $1\leq m\leq kr$). Les q_n satisfont à $(q_m, q_n)=\int_a^b q_mq_nd\alpha(x)=0$ pour $m\neq n\pm jr$. Ils constituent une généralisation des polynomes de Riesz pour lesquels la forme (2) est réduite à 2 termes. Le théorème de M. Favard est étendu aux polynomes q_n, ainsi que le fait que trois q_n successifs satisfont à une relation de récurrence à trois termes.

R. Campbell (Caen).

Lenz, Hanfried. Über die elliptischen Funktionen von Jacobi. Math. Z. **67** (1957), 153–175.

Désirant réaliser de la théorie des fonctions elliptiques une présentation qui tienne compte des remarques de Georg Faber [Burkhardt, Funktiontheoretische Vorlesungen, Bd. 2, 3. Aufl., de Gruyter, Berlin-Leipzig, 1920, p. VI], l'auteur propose un exposé de cette question qui simplifie encore notablement les méthodes employées par E. H. Neville dans "Jacobian elliptic functions" [Oxford, 1951; MR **13**, 24] en démontrant, pour commencer, une propriété de l'intégrale $\int (z(1-z)(1-\lambda z))^{-\frac{1}{2}}dz$ prise le long d'un contour triangulaire.

R. Campbell.

★ **Myrberg, P. J. Über automorphe Funktionen.** Proceedings of the International Congress of Mathematicians, 1954, Amsterdam, vol. III, pp. 118–126. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam, 1956. $7.00.

The author constructs automorphic functions for a given fuchsian group of real non-parabolic substitutions $S(z)=S=(az+b)/(cz+d)$, without using Poincaré theta functions. His basic function is of type

$$g(S)=\exp(\lambda_S u(z)+\eta_S)g(z),$$

where $u(z)$ is entire. He constructs primitive functions with given zeros and poles and obtains a representation of product type for an arbitrary automorphic function. As a special application, the $u(z)$ can be constructed directly by using the elliptic functions belonging to an auxilliary Riemann surface of genus one whose branch points occur in multiplicities which are integral divisors of those of the original surface. Details are omitted. *Harvey Cohn.*

★ **Maass, Hans. Differentialgleichungen und automorphe Funktionen.** Proceedings of the International Congress of Mathematicians, 1954, Amsterdam, vol. III, pp. 34–39. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam, 1956. $7.00.

The author surveys the development of differential equations in the theory of modular functions, starting with the observation that certain Eisenstein-type series are eigenfunctions of a suitable wave equation, and leading to matrix generalizations of complex variables [Maass, Math. Ann. **126** (1953), 44–68; MR **16**, 449]. The domains of such variables are interpreted in terms of E. Cartan's bounded symmetric spaces. *Harvey Cohn.*

Beckmann, Peter; und Franz, Walter. Asymptotisches Verhalten der Zylinderfunktionen in Abhängigkeit vom komplexen Index. Z. Angew. Math. Mech. **37** (1957), 17–27. (English, French and Russian summaries)

Die Sommerfeldsche Integraldarstellung der Zylinderfunktionen $Z_\nu(\rho)$ wird für festes reelles oder komplexes ρ in der komplexen ν-Ebene für ν-Werte von genügend großem Betrag nach der Sattelpunktsmethode untersucht. Die asymptotischen Gebiete in der ν-Ebene mit verschiedenem asymptotischen Verhalten der Hankel- oder Besselfunktionen werden bestimmt und die asymptotischen Darstellungen dieser Funktionen angegeben.

J. Meixner (Aachen).

Siegel, Keeve M. Bounds of the Legendre function. J. Math. Phys. **34** (1955), 43–49.

Es sei $P_n(x)$ die Legendresche Funktion der Ordnung n, $P_n'=dP_n/dx$, und es sei n_i $(i=0, 1, 2, \cdots)$ die Folge reeller positiver Zahlen, welche dadurch bestimmt ist, dass für ein gegebenes x_0, $-1<x_0<1$, die Gleichung $P_{n_i}'(x_0)=0$ besteht. Dann lässt sich zeigen, dass für $x_0\to -1$ die Beziehung $\lim_{x_0\to -1} P_{n_i}(x_0)\neq\infty$ besteht. Diese Tatsache ist von Bedeutung für die Behandlung von Randwertaufgaben der Schwingungsgleichung für kegelförmige Berandungen. Die Arbeit enthält ferner eine Zusammenstellung von Ungleichungen für die Werte von $P_{n_i}(x_0)$ und $P_{m_i}'(x_0)$, wobei m_i durch $P_{m_i}(x_0)=0$ bestimmt ist. *W. Magnus* (Zbl. **67** (1957), 46).

Schäfke, Friedrich Wilhelm. Integrale über Produkte von Sphäroidfunktionen. Math. Z. **67** (1957), 238–251.

Les fonctions sphéroïdales appelées quelquefois aussi "de Mathieu associées" généralisent celles de Mathieu pour lesquelles les formules relatives aux produits de 2 éléments sont déjà très compliquées.

Ce sont les fonctions propres de l'équation

$$\frac{d}{dz}\left[(1-z^2)\frac{dy}{dz}\right]+\left[\lambda+\gamma^2(1-z^2)-\frac{\mu^2}{1-z^2}\right]y(z)=0.$$

Utilisant ici la combinaison classique donnant naissance au wronskien, l'auteur fait apparaitre la quantité $\int_a^b y_1(z)y_2(z)(z^2-1)^{-1}dz$, et en déduit des formules où figure le produit de 2 fonctions sphéroïdales. (Pour une valeur particulière du paramètre, la formule est valable aussi pour les fonctions de Mathieu ordinaires.) Il en déduit une évaluation intéressante de $\partial\lambda/\partial\nu$ (ν étant l'indice de la fonction considérée qui est supposée pouvoir prendre des valeurs réelles quelconques).

Une méthode semblable avait été utilisée par Whittaker pour le même problème mais relatif aux fonctions de Bessel et aux fonctions sphériques. Dans ces derniers cas, on peut déduire des résultats obtenus les formules de récurrence habituelles de ces fonctions. Ainsi l'auteur repose-t-il très opportunément à cette occasion la vieille question de savoir si de semblables relations de récurrence peuvent exister entre des fonctions de Mathieu ou des fonctions sphéroïdales. Il est intéressant de remarquer en effet que les relations de récurrence obtenues pour les fonctions de Legendre et de Bessel ne se présentent à la fin de ce calcul que parce que les noyaux obtenus dans les intégrales sont dégénérés. Quand ils ne le sont pas, comme c'est le cas ici, de semblables relations n'apparaissent pas, et les remarques finales de cette étude concluent clairement que l'on ne peut espérer trouver entre trois fonctions de ce type une relation de récurrence que si les coefficients figurant dans la relation sont eux-mêmes exprimés par des fonctions de cette même classe (ce qui manque d'intérêt), et que d'ailleurs toute recherche relative à une simplification de leur expression serait illusoire.

R. Campbell (Caen).

Weisner, Louis. Group-theoretic origin of certain generating functions. Pacific J. Math. **5** (1955), 1033–1039.

Generating functions $g(x, y)=\sum_n g_n(x)y^n$ are considered. Expressions for $g(x, y)$ are given when $g_n(x)$ is a hypergeometric function, a Kummer function, or an ultraspherical polynomial. *F. Oberhettinger.*

See also: Analytic Theory of Numbers: Maass. **Functions of Complex Variables:** Gundlach. **Approximations, Orthogonal Functions:** Campbell. **Integral Transforms.** Zemanian; Rooney; Horváth. **Ordinary Differential Equations:** Thorne. **Topological Vector Spaces:** Gel'fand. **Complex Manifolds:** Gunning.

Sequences, Series, Summability

Green, H. F. The existence of bounds for infinite matrices. J. London Math. Soc. **32** (1957), 203–213.

If $\mathfrak{F}$ is a class of infinite matrices which contains all scalar matrices and is closed under finite sum and finite product, a bound is said to be defined on $\mathfrak{F}$ if with each matrix A belonging to $\mathfrak{F}$ we can associate a non-negative number $|A|$ satisfying the following conditions: if $A=(a_{i,j})$ and B belong to $\mathfrak{F}$, c is any scalar, and I is the unit matrix, then (i) $|cA|=|c||A|$, $|I|=1$, (ii) $|A+B|\leq|A|+|B|$, (iii) $|AB|\leq|A||B|$, (iv) $|a_{i,j}|\leq|A|$ $(i, j=1, 2, \cdots)$. The bound is said to be normal if, when $|b_{i,j}|\leq|a_{i,j}|$ for every i and j, and A is in the class $\mathfrak{F}$ on which the bound is defined, then $B=(b_{i,j})$ is in $\mathfrak{F}$, and $|B|\leq|A|$. A class $\mathfrak{F}$ is said to have the property N if, whenever A is in $\mathfrak{F}$ and $|b_{i,j}|\leq|a_{i,j}|$ for every i and j, then B is in $\mathfrak{F}$. A sequence space α is normal if, when x is in α and $|y_k|\leq|x_k|$ for all k, then y is in α.

The following results are proved. Th. 1: If the class $\mathfrak{F}$ of infinite matrices has the property N, then there exist normal sequence spaces α, β such that (i) $\alpha\supseteq\phi$, (ii) $\phi\subseteq\beta\subseteq\alpha^*$, (iii) $\mathfrak{F}\subseteq\sum_\beta(\alpha)$, where ϕ is the space of all finite

sequences. Th. 2: If a normal bound is defined on a maximum matrix ring R, then there exists a perfect sequence space α such that $R=\Sigma(\alpha)$. Th. 3: If R is a normal algebra of infinite matrices which contains M_f, the space of all matrices which are both row-bounded and column-bounded, and if R has the property N, then a norm $n(A)$ exists on R such that, if $A=(a_{i,j})\in R$ and $|b_{i,j}|\leqq|a_{i,j}|$ $(i,j=1,\ 2,\ \cdots)$, then $B=(b_{i,j})\in R$ and $n(B)\leqq n(A)$. Th. 4: If α is a perfect sequence space, a norm can be defined on the algebra $\Sigma(\alpha)$ if, and only if, α is a Köthe-Banach space. (A perfect sequence space which is a Banach space and in which distance convergence and strong projective convergence coincide is called a Köthe-Banach space.) Th. 5: If (i) α is perfect, (ii) $\sigma_1\subseteq\alpha\subseteq\sigma_\infty$, (iii) a norm $|A|$ is defined on $\Sigma(\alpha)$, then there exists $k\geqq1$ such that $|a_{i,j}|\leqq k|A|$ for every i and j, and every $A\in\Sigma(\alpha)$; k is independent of A. (σ_∞ is the space of all bounded sequences, and σ_1 that of all $\{x_k\}$ such that $\sum|x_k|$ converges.) Th. 6: If (i) α is perfect and $\sigma_1\subseteq\alpha\subseteq\sigma_\infty$, (ii) $\Sigma(\alpha)$ is a normed algebra, then a norm $n(x)$ exists on the vector space α with the properties (a) $n(e^{(i)})=1$ for each unit vector $e^{(i)}$ $(i=1,2,\cdots)$, (b) if $y=\{y_i\}\in\alpha$ and $|x_i|\leq|y_i|$ $(i=1,\ 2,\ \cdots)$, then $n(x)\leqq n(y)$, (c) $|x_i|\leqq n(x)$ $(i=1,2,\cdots)$ for every $x=\{x_i\}\in\alpha$, (d) α is a Köthe-Banach space with respect to the norm $n(x)$. Th. 7: If α is a perfect sequence space, a bound exists on $\Sigma(\alpha)$ if, and only if, (i) α is a Köthe-Banach space, (ii) $\sigma_1\subseteq\alpha\subseteq\sigma_\infty$. Th. 8: If (i) α is perfect and $\sigma_1\subseteq\alpha\subseteq\sigma_\infty$, (ii) $\Sigma(\alpha)$ is a normed algebra, then a bound exists on $\Sigma(\alpha)$. If, further, (iii) $\Sigma(\alpha)$ has the property N, then a normal bound exists on $\Sigma(\alpha)$. Th. 9: If (i) α and α^* (the dual space of α) are separable Köthe-Banach spaces, (ii) $\sigma_1\subseteq\alpha\subseteq\sigma_\infty$, then a semi-closed bound exists on $\Sigma(\alpha)$. Th. 10: If a normal semi-closed bound is defined on a class $\mathfrak{F}$ which contains M_f, then the bound is not semi-closed with respect to any proper subclass of $\mathfrak{F}$ which contains M_f. Cor.: If $\alpha\supseteq\phi$ and $\Sigma(\alpha)$ is a ring, then a normal semi-closed bound on $\Sigma(\alpha)$ is not semi-closed with respect to any subring of the type $\Sigma_\beta(\alpha)$, if $\beta\neq\alpha^*$. Th. 11: If F_r $(r>0)$ is the space of all sequences $\{x_k\}$ such that $\sum_{k=1}^\infty k^r|x_k|$ converges, then, under suitable definitions, $\Sigma(F_r)$ $(r>0)$ is a normed algebra on which no bound exists. [For terminology not already explained above, and for general background of this paper, see the reviewer's "Infinite matrices and sequence spaces", Macmillan, London, 1950, Ch. 2, 10; and "Linear operators", ibid., 1953, Ch. 6; MR 12, 694; 15, 719]. *R. G. Cooke* (London).

Vučković, Vladeta. Ein Satz über reelle Folgen. Riv. Mat. Univ. Parma 7 (1956), 139–140.

Let $a_1, a_2, \cdots$ and $b_1, b_2, \cdots$ be two real sequences such that $b_n>a_n$ whenever $a_n>a_{n-1}$, $b_n=a_n$ whenever $a_n=a_{n-1}$, and $b_n<a_n$ whenever $a_n<a_{n-1}$. If there is no integer N such that the sequence $a_N, a_{N+1}, \cdots$ is monotone, then

$$\liminf b_n\leqq\liminf a_n\leqq\limsup a_n\leqq\limsup b_n.$$

This theorem is a revision of the statement of the paper. An example (for which $a_1=1$, $a_n=1-n^{-1}$ when $n>1$, and $b_n=2$ when $n>2$) shows that the hypothesis involving monotonicity cannot be reduced to the hypothesis (the one inadvertently given in the paper) that the sequence $a_1, a_2, \cdots$ is not monotone. To prove the last part of the theorem, we observe that if there is a number y for which

$$\limsup b_n<y<\limsup a_n,$$

then there is an integer N such that $b_n<y$ when $n\geqq N$ and, moreover, there is an infinite set E of integers n for which $n\geqq N$ and $a_n>y$. If n is in E and $n\geqq N+1$, then $b_n<a_n$ so $a_{n-1}>a_n>y$ and $n-1$ is in E. This implies that E contains all of the integers N, $N+1$, $N+2$, $\cdots$ and that the sequence $a_N, a_{N+1}, \cdots$ must be monotone decreasing. Proof of the first part is similar. *R. P. Agnew.*

Kennedy, P. B. Integrability theorems for power series. Quart. J. Math., Oxford Ser. (2) 6 (1955), 316–320.

Heywood proved [J. London Math. Soc. 30 (1955), 302–310; MR 16, 1100] that if $c_n\geqq0$ for all sufficiently large n, and if (1) $\sum_0^\infty c_n=0$, then (2) $(1-x)^{-1}\sum_0^\infty c_nx^n\in L(0,1)$ if and only if (3) $\sum c_n\log n$ converges. The main object of this note is to obtain sufficient conditions for the equivalence of (2) and (3) when c_n is not assumed to be ultimately non-negative. First it is observed that (1) and (3) imply $s_n\log n\to0$ where $s_n=\sum_0^n c_r$. Theorem I: Any two of the statements (3), (4), (5) imply the remaining one; here (4), (5) read as follows: (4) $s_n\log n$ tends to a finite limit, (5) $\int_0^{1-\varepsilon}[(1-x)^{-1}\sum_0^\infty c_nx^n]dx$ tends to a finite limit as $\varepsilon\to+0$. Theorem II. Suppose there is an integer $p\geqq0$ such that, for all sufficiently large n, the p times iterated sums $S_n^{(p)}$ of the c's are $\geqq0$. Then any two of the statements (2), (3), (4) imply the remaining one.

B. Sz.-Nagy (Zbl. 66 (1957), 312).

Yu, Chia-Yung. On generalized Dirichlet series. Acta Math. Sinica 5 (1955), 295–311. (Chinese. English summary)

In the first of four parts of this article a theorem of Mandelbrojt [Séries adhérentes, régularisation des suites, applications, Paris, 1952, pp. 92–94; MR 14, 542] on Dirichlet series is generalized. In Part 2, half-planes of convergence are given for certain Dirichlet series. In Part 3, the preceding results are used to detect singularities of the functions represented by certain generalized Dirichlet series. Part 4 gives results on the growth of entire functions defined by certain generalized Dirichlet series.

From the author's summary.

Wynn, P. A note on Salzer's method for summing certain slowly convergent series. J. Math. Phys. 35 (1956), 318–320.

Approximations $S_{n,m}$ to the value S of a convergent series $\sum u_k$ with partial sums $S_1, S_2, \cdots$ are obtained by considering S_k to be the value of a function $f(x)$ when $x=1/k$, employing an interpolation formula to obtain the polynomial $P_{n,m}(x)$ of least degree which agrees with $f(x)$ when $x=1/n$, $1/(n+1)$, $\cdots$, $1/(n+m-1)$, and then putting $S_{n,m}=P_{n,m}(0)$. The convenient formula

$$S_{n,m}=S_n+\sum_{k=0}^{m-2}\frac{(n-1)!(n+k+1)}{(n+k-1)!(k+1)!}u_{n+k+1}$$

is obtained. Taking $n=m=4$ suffices to give, correct to 5 or 6 D, the value $\pi^2/6$ of the series $\sum 1/k^2$ and the value $\log 2$ of the series $(1-\frac{1}{2})+(\frac{1}{3}-\frac{1}{4})+(\frac{1}{5}-\frac{1}{6})+\cdots$.

R. P. Agnew (Ithaca, N.Y.).

Roberts, J. B. Matrix summability in F-fields. Proc. Amer. Math. Soc. 8 (1957), 541–543.

Some aspects are considered of summability theory in F-fields, i.e., in fields which are complete with respect to a non-Archimedean valuation. The infinite matrix $A=(a_{i,j})$ will be associated with the

sequence to sequence
series to sequence
series to series } transformation $\begin{cases}[u_n]\to[u_n'] \\ \sum_{i=0}^\infty u_i\to[u_n'] \\ \sum_{i=0}^\infty u_i\to\sum_{i=0}^\infty u_i'\end{cases}$

when $u_n'=\sum_{i=0}^{\infty} a_{i,n}u_i$ for all $n\geqq 0$. If the sequence to sequence transformation associated with A is such that $[u_n']$ exists and converges whenever $[u_n]$ converges, we call A a K matrix. If $[u_n']$ exists and converges to $\lim_{n\to\infty} u_n$ whenever this limit exists, we call A a T matrix. Similar names for the series to sequence and the series to series matrices are K_1, T_1 and K_2, T_2 matrices respectively. (K_1, T_1, T_2 matrices are sometimes referred to as β, γ, α matrices, respectively). The symbol $|a|$ is used for the valuation of a in the F-field under consideration. Consider the following three conditions: (i) $|a_{i,j}|<H$ for $i, j\geqq 0$ and some real H; (ii) $\sum_{i=0}^{\infty} a_{i,j}$ exists for all j and tends to ν as $j\to\infty$; (iii) for $i\geqq 0$, $\lim_{j\to\infty} a_{i,j}$ exists and equals α_i. Then the following theorem has been established by the author [Univ. of Minnesota Thesis, 1955]: The matrix $A=(a_{i,j})$ is a (1) K matrix if and only if (i), (ii), (iii); (2) T matrix if and only if (i), (ii), (iii), and $\nu=1$, and all $\alpha_i=0$; (3) K_1 matrix if and only if (i), (iii); (4) T_1 matrix if and only if (i),(iii) and all $\alpha_i=1$; (5) K_2 matrix if and only if (i) and $\sum_{j=0}^{\infty} a_{i,j}$ exists for $i\geqq 0$; (6) T_2 matrix if and only if (i) and $\sum_{j=0}^{\infty} a_{i,j}=1$ for $i\geqq 0$. Furthermore if A is a (a) K matrix, then $u_n'\to \nu u+\sum_{i=0}^{\infty}\alpha_i(u_i-u)$; (b) K_1 matrix, then $u_n'\to \alpha_0\sum_{i=0}^{\infty} u_i+\sum_{i=0}^{\infty}\{(\alpha_{i+1}-\alpha_i)\sum_{j=i+1}^{\infty} u_j\}$; here $u=\lim_{i\to\infty} u_i$. The only difficulty in the proofs is the proof of the necessity of (i) in each of (1)–(6). The object of this note is to give a brief proof of this by proving a "key lemma" based on F-Banach spaces.

A vector space S over the F-field F, with valuation $|\cdot|$, is an F-Banach space if there exists a real-valued function $\|\cdot\|$ satisfying (i) $\|x\|$ exists and is $\geqq 0$ for all $x\in S$, and $\|x\|=0$ only for $x=0$; (ii) $\|x+y\|\leqq\|x\|+\|y\|$, $x\in S$, $y\in S$; (iii) $\|cx\|=|c|\cdot\|x\|$ for all $c\in F$ and $x\in S$; (iv) S is complete with respect to $\|\cdot\|$. Let (m) and (c_0) denote the sets of bounded and null sequences respectively; both (m) and (c_0) are F-Banach spaces if we define $\|x\|=\sup_n |x_n|$ for $x=[x_n]\in(m)$. The "key lemma" is then as follows. If, for arbitrary $[x_n]\in(c_0)$, $x_n'=\sum_{i=0}^{\infty} a_{i,n}x_i$ exists for all $n\geqq 0$ and $[x_n']\in(m)$, then there exists a real number $H>0$ such that $|a_{i,j}|<H$ for $i, j\geqq 0$. The proof is based on an analogue of the Banach-Steinhaus theorem for F-Banach spaces. *R. G. Cooke* (London).

Flett, T. M. On an extension of absolute summability and some theorems of Littlewood and Paley. Proc. London Math. Soc. (3) **7** (1957), 113–141.

Let $k\geqq 1$. A series $\sum a_n$ is said to be evaluable $|A|_k$, or to be absolutely evaluable by the Abel method having index k, if the series $\sum a_n x^n$ converges over $0\leqq x<1$ to a finite $\phi(x)$ for which

$$\int_0^1 (1-x)^{k-1}|\phi'(x)|^k dx<\infty.$$

When $k\geqq 1$ and $\alpha>-1$, the series is evaluable $|C,\alpha|_k$ if

$$\sum n^{k-1}|\sigma_n^{(\alpha)}-\sigma_{n-1}^{(\alpha)}|<\infty,$$

where $\sigma_1^{(\alpha)}$, $\sigma_2^{(\alpha)}$, $\cdots$ is the Cesàro transform of order α of the series. Consequences of these definitions are systematically and extensively investigated, and it is shown that these definitions constitute a very reasonable extension to the whole set $k\geqq 1$ of the familiar definitions to which they reduce when $k=1$. Many of the results which are known for the case $k=1$ are generalized to the case $k\geqq 1$. Many sharp inequalities are obtained. Integrals such as

$$\left\{\int_0^1 (1-\rho)^{k-1}|\phi'(\rho e^{i\theta})|^k d\rho\right\}^{1/k},$$

$$\left\{\int_0^{2\pi} (1-\rho)^{k-1}|\phi'(\rho e^{i\theta})|^k d\theta\right\}^{1/k}$$

are studied and function-theoretic applications are obtained. In addition to new results for the case $k>1$, the paper provides a useful source of information and references for the case $k=1$. *R. P. Agnew* (Ithaca, N.Y.).

Flett, T. M. Some theorems on power series. Proc. London Math. Soc. (3) **7** (1957), 211–218.

This is a sequel to the paper reviewed above. Alternative proofs of several theorems and inequalities are given. Some additional theorems involving evaluability, absolute evaluability, and strong evaluability (by the Abel and Cesàro methods) are given. Some of these are Abelian and others are Tauberian. Most of them are consequences and analogues of the following theorem. If $r\geqq k\geqq 1$, $\alpha>-1$, $\beta\geqq\alpha+1/k-1/r$, then there is a constant $A=A(k, r, \alpha, \beta)$ such that

$$\left\{\sum_{n=0}^{\infty}\frac{|\sigma_n^{(\beta)}-s|^r}{n+1}\right\}^{1/r}\leqq A\left\{\sum_{n=0}^{\infty}\frac{|\sigma_n^{(\alpha)}-s|^k}{n+1}\right\}^{1/k}$$

whenever $\sum a_n$ is a series and s is a constant for which the right member is finite. *R. P. Agnew* (Ithaca, N.Y.).

Flett, T. M. A high-indices theorem. Proc. London Math. Soc. (3) **7** (1957), 142–149.

The following Tauberian gap theorem, which has function-theoretic applications, is proved. Let $c>1$ and $k\geqq 1$. Then there is a constant $A(c, k)$, depending only upon c and k, such that

$$\sum_{n=1}^{\infty}|a_n|^k\leqq A(c, k)\int_0^1 (1-x)^{k-1}|\phi'(x)|^k dx$$

whenever (i) the series $\sum a_n$ is lacunary in the sense that there is a sequence $n_1, n_2, \cdots$ of positive integers such that $n_{p+1}/n_p>c$ and $a_n=0$ whenever $n\neq n_1, n_2, \cdots$; (ii) the series in $\phi(x)=\sum a_n x^n$ converges when $0\leqq x<1$; and (iii) the right member of the inequality is finite. This theorem on power series is a consequence of the following more general theorem on Dirichlet series in which $\lambda_1, \lambda_2, \cdots$ are not required to be integers. Let $c>1$ and $k\geqq 1$. Then there is a constant $A(c, k)$ such that

$$\sum_{n=1}^{\infty}|a_n|^k\leqq A(c, k)\int_0^{\infty}(e^s-1)^{k-1}|f'(s)|^k ds$$

whenever (1) $\lambda_1>0$ and $\lambda_{n+1}/\lambda_n>c$ for each n; (2) the series in

$$f(s)=\sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$$

converges when $s>0$; and (3) the right member of the inequality is finite. The proof of this, like the Zygmund proof [Trans. Amer. Math. Soc. **55** (1944), 170–204; MR **5**, 230] for the case $k=1$, is based upon a method of Ingham [Quart. J. Math. Oxford Ser. **8** (1937), 1–7] but the details here are much more troublesome. *R. P. Agnew.*

★ **Obreschkoff, Nikola. Asymptotische Formeln zur angenäherten Auswertung von Summen unendlicher Reihen.** Aktuelle Probleme der Rechentechnik. Bericht über das Internationale Mathematiker-Kolloquium, Dresden, 22. bis 27. November 1955, pp. 119–125. VEB Deutscher Verlag der Wissenschaften, Berlin, 1957.

In the present paper the author is concerned with obtaining estimates of the remainder, R_n, after n terms of the infinite series $u_1+u_2+\cdots$. A number of theorems are proven of which the following is typical. Suppose, for $n>n_0$, $u_{n+1}/u_n=x(1+\alpha_n/(n+\lambda))$, where λ is a constant

$|x|<1$ and $\lim \alpha_n=\alpha$; then for $n>n_0$,

$$R_n=u_n\left(\frac{x}{1-x}+\frac{\theta_n\alpha x}{(n+\lambda)(1-x)^2}\right),$$

where $\lim \theta_n=1$. *V. F. Cowling* (Lexington, Ky.).

Pitt, H. R. An elementary proof of the closure in L of translations of e^{-x^2}, and the Borel Tauberian theorem. Proc. Amer. Math. Soc. **8** (1957), 706–707.

The Hardy-Littlewood theorem for Borel summability asserts that the relations $e^{-t}\sum s_n t^n/n!\to 0$ as $t\to\infty$ and $a_n=O(n^{-1})$ imply that $s_n=\sum_{k\le n} a_k\to 0$. It follows from the hypotheses of the theorem that $s(x)=\sum_{k\le x} a_k$ is bounded and that $\int_{-\infty}^{\infty} e^{-(u-y)^2}s(\frac{1}{2}u^2)du\to 0$ as $y\to\infty$. [Hardy, Divergent series, Oxford, 1949, pp. 313, 314; MR **11**, 25]. Thus the conclusion $s(x)\to 0$ becomes a relatively simple corollary of the theorem which states that the translations of e^{-x^2} form a topologically generating subset (fundamental subset) of $L_1(-\infty,\infty)$. The author gives a short direct proof of this closure theorem.

J. Korevaar (Madison, Wis.).

Approximations, Orthogonal Functions

Zuhovickiĭ, S. I.; and Stečkin, S. B. On approximation of abstract functions. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 1(73), 187–191. (Russian)

Summary and discussion of results announced earlier [Dokl. Akad. Nauk SSSR (N.S.) **106** (1956), 385–388, 773–776; MR **18**, 222; **19**, 30]. *E. Hewitt.*

Vertgeĭm, B. A. On certain methods of approximate solution of non-linear functional equations in Banach spaces. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 1(73), 166–169. (Russian)

The author states, without proof, conditions for applicability of Newton's method, as extended by L. V. Kantorovič [same Uspehi **3** (1948), no. 6(28), 89–185; Trudy Mat. Inst. Steklov **28** (1949), 104–144; MR **10**, 380; **14**, 766; **12**, 419], for approximate solution of operator equations in Banach spaces. [See also, Dokl. Akad. Nauk SSSR (N.S.) **110** (1956), 719–722; MR **18**, 734.] The note also includes short discussions of the effect of round-off error on convergence of the sequence of successive approximations obtained by Newton's method and of application of the method to construction of conformal maps. *M. Jerison* (Lafayette, Ind.).

Hummel, J. A. Complete orthonormal sequences of functions uniformly small on a subset. Proc. Amer. Math. Soc. **8** (1957), 492–495.

The author proves the following theorem: Let H_1 and H_2 be two separable Hilbert spaces and let $J: H_2\to H_1$ be a completely continuous linear mapping. Then for any $\varepsilon>0$ there is a complete orthonormal sequence $\{\varphi_n\}$ such that $\|J\varphi_n\|<\varepsilon$ for all n. The proof consists of applying an N-dimensional unitary transformation whose terms are of equal modulus to a sufficiently long intitial segment of any orthonormal sequence in H_1 (there exist such unitary transformations for arbitrarily large N and in fact for all N).

When applied to $H_1=L^2(D')$, $H_2=L^2(D)$, where $\bar{D}'=K\subset D$, D and D' are bounded domains in the z-plane $L^2(D)$ is the Hilbert space of analytic functions in D such that $(f,g)=\iint f\bar{g}dxdy$, and J is the restriction mapping, this yields an orthonormal sequence in $L^2(D)$ whose terms are uniformly bounded by ε throughout the compact set K. In particular, for any N, K, $\varepsilon>0$, there is an orthonormal sequence $\{\varphi_n\}$ in $L^2(D)$ such that, for all z in K, $\sum_1^N |\varphi_n(z)|^2/\sum_1^\infty |\varphi_n(z)|^2<\varepsilon$ for any re-arrangement of the terms. The author proves this but states in the text only the much weaker (and in fact trivial) version of the above for a fixed ordering, which can be proved by suitable re-arrangement of any orthonormal sequence.

A. B. Novikoff (Berkeley, Calif.).

Campbell, Robert. Généralisation de la formule de Fejér pour les séries de polynomes orthogonaux usuels. Ann. Sci. Ecole Norm. Sup. (3) **71** (1954), 389–419.

Let $\{P_n(x)\}_0^\infty$ be a system of polynomials, orthogonal on $[a,b]$ with the weight-function $p(x)$. Then it is well known that

$$(1)\quad P_n(x)=(A_nx+B_n)P_{n-1}(x)-A_nA_{n-1}^{-1}P_{n-2}(x)$$

and if a_n are the Fourier coefficients of the function $f(x)$, then

$$S_n[f;x]=S_n(x)=\sum_0^n a_kP_k(x)=A_{n+1}^{-1}\int_a^b G_n(x,t)(x-t)^{-1}f(t)p(t)dt,$$

where $G_n(x,t)=P_{n+1}(x)P_n(t)-P_n(x)P_{n+1}(t)$.

Setting $\sigma_n^{(1)}(x)=n^{-1}\sum_0^{n-1} S_k(x)$ the author, under certain hypotheses, seeks an explicit expression for $\sigma_n^{(1)}$ in the form of a singular integral. Namely, beginning with the relation

$$\sum_0^{n-1} A_{k+1}^{-1}G_k(x,t)=-A_n^{-1}P_{n-1}(x)P_n(t)+\sum_1^{n-1}P_k(t)[A_{k+1}^{-1}P_{k+1}(x)-A_k^{-1}P_{k-1}(x)]+A_1^{-1}P_1(x)P_0(t)$$

and introducing the functions $f_n(x)$ and $g_n(x)$, for which $P_n(x)=f_n(x)P_{n-1}(x)-g_n(x)P_{n-1}'(x)$, it is easy to obtain

$$\sum_0^{n-1} A_{k+1}^{-1}G_k(x,t)=A_n^{-1}P_{n-1}(x)P_n(t)+\sum_1^{n-1}P_k(t)[U_k(x)P_k(x)-V_k(x)P_k'(x)]+A_1^{-1}P_1(x)P_0(t),$$

where

$$U_k(x)=2A_{k+1}^{-1}f_{k+1}(x)-x-B_{k+1}A_{k+1}^{-1},\qquad V_k(x)=2A_{k+1}^{-1}g_{k+1}(x).$$

Assume that it is possible to choose $f_n(x)$ and $g_n(x)$ in such a way that $U_k(x)$ and $V_k(x)$ do not depend on k. Then, making use of the relation, deducible from (1),

$$(x-t)P_n(t)P_n'(x)=\Delta_{n+1}(x,t)-\Delta_n(x,t)-P_n(t)P_n(x),$$

where $\Delta_n(x,t)$ is given by

$$\Delta_n(x,t)=\frac{1}{A_n}\begin{vmatrix}P_n'(x) & P_{n-1}'(x)\\ P_n(t) & P_{n-1}(t)\end{vmatrix},$$

and also using the Christoffel-Darboux formula

$$\sum_0^n P_k(x)P_k(t)=A_{n+1}^{-1}G_n(x,t)/(x-t),$$

it is possible to find an expression for $\sigma_n^{(1)}$ in finite form.

In particular, this method enables us to find an expression for $\sigma_n^{(1)}$ in the case of Fourier series in Hermite and Laguerre polynomials.

Moreover, the author considers generalized means $m_n=\sum_0^n \alpha_k S_k(x)/\sum_0^n \alpha_k$, where the constants α_k are such that the corresponding functions $U_k(x)$ and $V_k(x)$ do not depend on k. By a method analogous to the above it is possible to find an explicit expression for m_n. In this way generalized means m_n are found for Fourier series in ultraspherical polynomials.

Finally the question is studied for Cesàro means and generalized means of higher orders and the method discussed is applied to series in Bessel functions.

{Reviewer's remark: The whole third chapter of the work is unconvincing, since the arguments for formulas (5) and (6) of this chapter are not conclusive. And in any case, when the author operates with asymptotic formulas, his arguments, in the opinion of the reviewer, are not correct, since he is not sufficiently careful in estimating the influence of various mathematical operations on the order of the remainder terms in these formulas (for example, on p. 401 in place of $o(1)/(x-t)$ he writes $O(1)$). There are many misprints and slight mistakes.}

G. I. Natanson (RŽ Mat 1956, no. 326).

Moser, Leo; and Wyman, Max. Asymptotic expansions. II. Canad. J. Math. **9** (1957), 194–209.

The authors continue their investigation [same J. **8** (1956), 225–233; MR **17**, 1201] of the coefficients b_n given by $\exp(P(x))=\sum b_n x^n/n!$, where $P(x)$ is a polynomial with arbitrary real coefficients, removing their previous restriction that the coefficients be non-negative.

S. Chowla (Princeton, N.J.).

Davis, Philip. Uniqueness theory for asymptotic expansions in general regions. Pacific J. Math. **7** (1957), 849–859.

Let D be a simply connected region with an analytic boundary C, $z=0$ an interior point and $z=1$ a boundary point. Let $f(z)$ be regular in D and possess the asymptotic expansion $\sum_0^\infty a_n(z-1)^n$. If

$$f_n(z)=(z-1)^{-n}[f(z)-\sum_0^{n-1} a_\nu(z-1)^\nu],$$

then, given a sequence $\{m_n\}$ of positive quantities, we consider functions for which $\|f_n(z)\|<Mk^n m_n^2$ ($n=0, 1, 2, \cdots$) for some $M>0$ and $k>0$. The uniqueness or non-uniqueness of the asymptotic expansion has been shown to depend on the growth of the m_n by Watson [Philos. Trans. Roy. Soc. London. Ser. A. **211** (1912), 279–313] and F. Nevanlinna [Ann. Acad. Sci. Fenn. Ser. A. **12** (1918), no. 3] when D is a sector; by Carleman [Les fonctions quasi analytiques, Gauthier-Villars, Paris, 1926] and Ostrowski [Acta Math. **53** (1929), 181–266] when D is a circle; by Mandelbrojt and MacLane [Trans. Amer. Math. Soc. **61** (1947), 454–467; MR **8**, 508] and by Meili [Comment. Math. Helv. **29** (1954), 93–96; MR **16**, 583] for more general regions. In all these cases $\|f(z)\| \equiv \max_{z\in D} |f(z)|$. The author deals with the uniqueness problem for the region D defined above, using the norm $\|f(z)\|^2 \equiv \int_C |f(z)|^2 ds$ and techniques derived from the theory of the kernel function and conformal mapping as developed by Bergman [The kernel function and conformal mapping, Math. Surveys, no. 5, Amer. Math. Soc., New York, 1950; MR **12**, 402] and Szegö [Orthogonal polynomials, Amer. Math. Soc. Colloq. Publ., v. 23, New York, 1939; MR **1**, 14]. This leads to a minimising process from which an elegant solution of the uniqueness problem arises in a natural way. The main result is as follows: Let $w=m(z)$ map D conformally onto the unit circle with $m(0)=0$ and $m(1)=1$. Then, given an arbitrary sequence of positive constants m_n, the class of $f(z)$ is a uniqueness class for asymptotic expansions at $z=1$ if and only if, for all $t>0$,

$$\limsup_{n\to\infty} \int_C \log\left\{\sum_{k=0}^n \frac{t^k}{m_k^2}|(z-1)^{n-k}|^2\right\}\frac{\partial}{\partial\nu}\log|m(z)|ds=\infty,$$

where $\partial/\partial\nu$ denotes normal differentiation in the positive sense.

R. Wilson (Swansea).

See also: Sequences, Series, Summability: Pitt. **Topological Vector Spaces:** Holladay.

Trigonometric Series and Integrals

★ **Chandrasekharan, K. On some problems in Fourier analysis.** Proceedings of the International Congress of Mathematicians, 1954, Amsterdam, vol. III, pp. 85–91. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam, 1956. $7.00.

The author presents a review of solved and unsolved problems in Fourier analysis. They concern the problem of summability methods with localization property, especially in several variables, of bounded convergence in L^p, of localization for trigonometric series in place of Fourier series, of uniqueness of trigonometric series and integrals, of generalized integrals which arise in the theory of uniqueness, and of corresponding results for eigenfunction expansions of the Sturm-Liouville problem in one and more dimensions.

František Wolf.

Ishiguro, Kazuo. Fourier series. XI. Gibbs' phenomenon. Kōdai Math. Sem. Rep. **8** (1956), 181–188.

When $f(x)$ has a discontinuity of a certain kind at $x=x_0$, there exists a number r_0, $0<r_0<1$, such that the (C, r) means of the Fourier series of $f(x)$ present Gibbs' phenomenon at x_0 for $r<r_0$, but not for $r\geqq r_0$. For a simple discontinuity, this was proved by the reviewer [Ark. Mat. Astr. Fys. **13** (1918), no. 20]. The author now proves the same theorem for more general classes of discontinuities. In a correction to be published, the author points out that the conditions of one of his theorems require certain modifications.

H. Cramér.

Kahane, Jean-Pierre; et Salem, Raphaël. Construction de pseudomesures sur les ensembles parfaits symétriques. C. R. Acad. Sci. Paris **243** (1956), 1986–1988.

This is the third paper of a series [cf. MR **18**, 651; **19**, 268] in which the authors study the spectral synthesis of a perfect set. The following terminology is used. If $E\subset(0, 2\pi)$ is a closed set and $\{c_n\}_{-\infty}^{\infty}$ a bounded sequence with its spectrum in E, $\sum c_n e^{inx}$, the Fourier series of a distribution with E as carrier, is called a pseudomeasure (in contrast to the case where $\sum c_n e^{inx}$ is a Fourier-Stieltjes series and represents a measure). E is called free of pure pseudomeasures if every bounded sequence with spectrum in E is a sequence of Fourier-Stieltjes coefficients. If E is a symmetric perfect set (i.e., the points of E are given by the formula $2\pi\sum \varepsilon_i r_i$ where ε is 0 or 1, $r_i>0$, $\sum_1^\infty r_i=1$ and $r_p>\sum_{p+1}^\infty r_k$), the authors show that E carries a pseudomeasure with a continuous primitive. One of the two methods given to obtain this result is as follows: $f(x)=\sum \varepsilon_p 2^{-p}$ if $x=2\pi\sum \varepsilon_p r_p$; $H_s=\int_0^x \sum_1^s G_i\,dy$, where $G_1(y)=g_1(y)$, $G_s=g_{m_s}g_{m_s+2}\cdots g_{m_s+2(s-1)}$ with $m_s=1+s(s-1)$ and $g_m(y)=\frac{1}{2}(\varphi_m+\varphi_{m+1}+\varphi_m\cdot\varphi_{m+1})$, where $\varphi_k(y)$ are the Rademacher functions; $f(x)$ is known to be

continuous (Lebesgue's function); $H_s(y)$ is continuous of bounded variation and tends uniformly to a continuous function $H(y)$ if s tends to ∞. They show that the Fourier coefficients of $H(f(x))$ are $O(1/n)$ and hence this function is a primitive of a pseudomeasure carried by E. Finally, they remark that if $r_k=\xi^k(\xi-1)$, $0<\xi<\frac{1}{2}$, i.e., if E is of constant ratio, the Fourier coefficients of $H(f(x))$ are $O(1/n)$. *W. A. J. Luxemburg* (Toronto, Ont.).

Kumari, Sulaxana. On the order of the Cesàro means of a series conjugate to Fourier series. Bull. Calcutta Math. Soc. **48** (1956), 139–151.

Let $\bar{S}_n(x)$ denote the partial sums of the conjugate series

$$\sum (b_n \cos n\theta - a_n \sin n\theta) = \sum B_n(\theta)$$

of the Fourier series corresponding to a function f (integrable, periodic). Lukács [J. Reine Angew. Math. **150** (1920), 107–112] proved $\bar{S}_n(x) \sim (2l/\pi) \log n$, if $2\cdot\theta(t) = f(x+t)-f(x-t)-2l \to 0$. The authoress extends this result to Cesàro-means of $\bar{S}_n$, using conditions on Cesàro-means of f. Theorem 1: If $\int_0^t |\theta_\alpha(t)|dt = o[(\log 1/t)^{p+1}]$, where $\theta_\alpha(t)=\alpha\cdot t^{-\alpha}\int_0^t (t-u)^{\alpha-1}\theta(u)du$ $(\alpha>0)$ and $\theta_0(t)=\theta(t)$, then the C_α-means $\bar{S}_\alpha(\omega)$ of the $\bar{S}_n$ satisfy $\bar{S}_\alpha(\omega) \sim (2l/\pi)\log\omega$ $(-1<p\leqq 0)$, $\bar{S}_\alpha(\omega)=o[(\log\omega)^{p+1}]$ $(p>0)$. Theorem 2 is the corresponding statement concerning the sequence $\{nB_n(x)\}$. Both theorems are best possible. Theorem 3 is the extension to differentiated series [cf. M. L. Misra, Bull. Calcutta Math. Soc. **38** (1946), 151–155; MR **8**, 577; Mohanty and Nanda, Proc. Amer. Math. Soc. **6** (1955), 594–597; MR **17**, 150]. Corollaries of the results are sufficient conditions for Riesz summability of the conjugate series and the derived series. In a paper in publication (no reference given) the authoress similarly treats the case of the Fourier series itself. *K. Zeller.*

Yano, Shigeki. On approximation by trigonometric polynomials. Kōdai Math. Sem. Rep. **8** (1956), 93–95.

The author proves the following theorem. Let $\{s_n(x)\}$ be a sequence of exponential polynomials, each $s_n(x)$ being at most of order n. If $(2\pi)^{-1}\int_0^{2\pi} |s_n(x)|dx \leqq M$, and if there exist two constants ε, $c>0$, and an interval (a, b) of length r, such that $|s_n(x)| \leqq ce^{-\varepsilon n}$ in (a, b), and if $\varepsilon r > 2\pi\rho$, where $\rho=\min_{0<t<\infty} 2t(e^t+1)/(e^t-1)$, then $\{s_n(x)\}$ converges uniformly to zero in the interval $-\pi\leqq x\leqq\pi$. This is the corrected form of Bochner's theorem on the localization of best approximation [Contributions to Fourier Analysis, Princeton, 1950, pp. 3–23; MR **12**, 255]. The need for a corrected version was first pointed out by M. E. Noble [Math. Ann. **128** (1954), 55–62, 256; MR **16**, 126]. The author states that his proof "is almost nothing but a repetition" of Bochner's original proof.

K. Chandrasekharan (Bombay).

Kinukawa, Masakiti. On the Riesz logarithmic summability of the conjugate derived Fourier series. I, II. Proc. Japan Acad. **31** (1955), 121–125, 202–206.

Let $f(x)=a_0/2+\sum_1^\infty A_n(x)$, $A_n(x)=a_n\cos nx+b_n\sin nx$, be a Fourier series and (*) $\sum_1^\infty nA_n(x)$ the conjugate derived series. The series $\sum_1^\infty a_n$ is summable $(R, \log, \alpha)$, to s, i.e., summable by Riesz's logarithmic mean of order $\alpha\geqq 0$, if

$$\lim_{\omega\to\infty} (\log\omega)^{-\alpha}\sum_{n<\omega} (\log\omega/n)^\alpha a_n = s.$$

Let

$$g(t)=\int_t^\infty [f(x+u)+f(x-u)-2s\cos u]\frac{du}{u^2},$$

$$g_\alpha(t)=(\log 1/t)^{-\alpha}\int_t^\pi (\log u/t)^{\alpha-1}g(u)\frac{du}{u};$$

then if $\lim_{t\to 0} g_\alpha(t)=0$, (*) is summable $(R, \log, \alpha+2)$ to S at x. If (*) is summable $(R, \log, \alpha+2)$ to S at x, then $\lim_{t\to 0} g_\beta(t)=0$ for $\beta>\alpha+1$. Another criterion for summability $(R, \log, \alpha+2)$ of (*) in terms of integrals in $g_\alpha(t)$ is given. *A. P. Calderón* (Cambridge, Mass.).

Musielak, J. Some conditions sufficient for the absolute convergence of multiple Fourier series. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 251–254, XXI–XXII. (Russian summary)

Let $f(x_1, x_2, \cdots, x_n)$ be an integrable function defined in the n-dimensional Euclidean space, with period 2π in each variable. Its Fourier coefficients are defined by

$$a^A_{m_1,\ldots,m_n}(f)=$$

$$\int_0^{2\pi}\cdots\int_0^{2\pi} f(x_1, \cdots, x_n) \prod_{i\in A}\cos m_i x_i \prod_{i\in \bar{A}}\sin m_i x_i dx_1\cdots dx_n,$$

where A is a subset of $(1, 2, \cdots, n)$, $\bar{A}$ its compliment and $m_1, m_2, \cdots$ are non-negative integers. The author gives sufficient conditions for convergence of the series

$$\sum_{m_1,m_2,\ldots,m_n=0}^{\infty} (m_1+1)^{\beta_1}(m_2+1)^{\beta_2}\cdots(m_n+1)^{\beta_n}\rho^{(\gamma)}_{m_1,m_2,\ldots,m_n}(f)$$

where $\beta_1, \beta_2, \cdots, \beta_n\geqq 0$, $0<\gamma<2$ and

$$\rho^{(\gamma)}_{m_1,\ldots,m_n}(f)=\sum_A |a^A_{m_1,\ldots,m_n}(f)|^\gamma,$$

summation being taken for all $A\subset(1, 2, \cdots, n)$. His conditions generalize results obtained for two variables by V. G. Chelidze [C. R. (Dokl.) Acad. Sci. URSS (N.S.) **54** (1946), 117–120; MR **8**, 376] and I. E. Žak [Soobšč. Akad. Nauk Gruzin. SSR **12** (1951), 129–133; MR **14**, 42].

S. Izumi (Chicago, Ill.).

Kestelman, H. Measurable almost periodic functions. Mathematika **3** (1956), 140–143.

In order to obtain an elementary proof of the known theorem (and also a somewhat stronger result) that a measurable function in $\mathscr{R}_n$ which is almost periodic in von Neumann's sense in $\mathscr{R}_n$ (here taken as a discrete group) is almost periodic in Bohr's sense, the author proves by elementary set theory that if f is any function which is measurable in $\mathscr{R}_n$ and has the property that for every $\varepsilon>0$ there is a set $\mathscr{E}_\varepsilon$ with positive exterior measure such that $|f(x+h)-f(y+h)|<\varepsilon$ for all x, y in $\mathscr{E}_\varepsilon$ and all h in $\mathscr{R}_n$, then f is uniformly continuous in $\mathscr{R}_n$. An elementary proof of the first stated theorem was previously given by H. D. Ursell in the case $n=1$ [J. London Math. Soc. **4** (1929), 123–127], "but it did not seem to be readily adaptable for $n>1$." *E. Følner.*

Upton, C. J. F. Riesz almost periodicity. J. London Math. Soc. **31** (1956), 407–426.

For $p=1$, E. R. Love [same J. **26** (1951), 14–25; MR **12**, 599] characterized the functions which are bounded integrals of $\mathscr{S}^p$-a.p. functions as the functions which in a certain norm, based on total variation, can be approximated by trigonometric polynomials. He also characterized these functions by translation properties.

The author generalizes this result to every $p \geqq 1$ and gives various other characterizations of the functions in question. The paper ends with a study of further classes of a.p. functions which possess higher degrees of differentiability. *E. Følner* (Copenhagen).

See also: Measure, Integration: Gavrilyuk; Taylor. **Integral Transforms:** Hewitt and Wigner. **Banach Spaces, Banach Algebras, Hilbert Spaces:** Krabbe. **Fluid Mechanics, Acoustics:** Sanoyan.

Integral Transforms

Hewitt, Edwin; and Wigner, Eugene P. On a theorem of Magnus. Proc. Amer. Math. Soc. **8** (1957), 740–744.

The results referred to in the title [same Proc. **6** (1955), 880–890; MR **17**, 362] are Fourier and Plancherel theorems for $n \times n$ matrices in which the inversion formulas involve integration over a suitable neighborhood of a covering space of the full unitary group. Roughly speaking, the present authors show that only the diagonal unitary matrices really count. Thus they obtain theorems which are easier to prove and which can immediately be generalized to locally compact abelian groups. *R. Steinberg.*

Watanabe, Yoshikatsu. Eine Verallgemeinerung der Laplace-Transformation. J. Gakugei Tokushima Univ. Nat. Sci. Math. **7** (1956), 19–35.

Expository article on double Laplace transforms.

R. P. Boas, Jr. (Evanston, Ill.).

Zemanian, Armen H. Some inequalities for Fourier transforms. Proc. Amer. Math. Soc. **8** (1957), 468–475.

Let $\phi(t)$ be real and integrable in $(0, \infty)$. Let

$$f(x)=\int_0^\infty \phi(t)\cos xt\,dt,\quad g(x)=\int_0^\infty \phi(t)\sin xt\,dt,$$

$$F(x)=\int_0^x f(y)\,dy.$$

The author establishes three inequalities. (1) If ϕ is nonincreasing and absolutely continuous and $|f(x)| \leqq kf(0)$, $0 \leqq k < 1$, then for $0 \leqq x < 1$

$$|f(x)| \leqq f(0)\left\{\frac{\sin \pi x}{\pi x}+k\left[\frac{\sin \pi x}{\pi x}-\cos \pi x.\right]\right\}.$$

(2) If $\phi \geqq 0$, $\int_0^\infty t\phi(t)\,dt=M$, and $|g(x)| \leqq k$, $k>0$, for $x \geqq 1$, then for $0<x<1$

$$\pi|g(x)| \leqq \sin \pi x\{M-k[\Psi(x)+\Psi(-x)+2\gamma]\},$$

where Ψ is the logarithmic derivative of Γ and γ is Euler's constant. (3) A similar but more complicated inequality for $F(x)$ when $\phi \geqq 0$ and ϕ is of bounded variation near 0. See also the review below. *R. P. Boas, Jr.*

Zemanian, Armen H. A property of Hurwitz polynomials and some associated inequalities for Fourier transforms. Proc. Amer. Math. Soc. **8** (1957), 716–723.

A Hurwitz polynomial is one whose roots have negative real parts. Let $Z(s)$ be the reciprocal of such a polynomial of degree m with leading coefficient 1, and let $R_k(t)$ and $I_k(t)$ be obtained by integrating the real and imaginary parts of $Z(it)$ $k(\leqq m-1)$ times with lower limit $-\infty$. Then R_k has the parity of k and I_k has the parity of $k-1$; when $m=2n+1$, R_k has $m-k-1$ simple zeros, I_k has $m-k$ simple zeros, and $(-1)^n R_{2n}(t)>0$ for all real t; when $m=2n$, R_k has $m-k$ simple zeros, I_k has $m-k-1$ simple zeros, and $(-1)^{n+1}I_{2n-1}(t)>0$. Let

$$f(x)=(2\pi)^{-1}\int_{-\infty}^{\infty} Z(it)e^{ixt}\,dt.$$

If $|f(x)| \leqq A$ for $x \geqq 1$, then for $0 \leqq x < 1$ we have

$$|f(x)| \leqq \frac{x^{m-2}\sin \pi x}{\pi(m-1)!}+2Ax^m\pi^{-1}\sin \pi x \sum_{p=1}^{\infty} p^{-m+1}/(p^2-x^2),$$

The cases $m=1$, 2 are contained in the paper reviewed above, with less restrictive hypotheses on f. A further theorem gives bounds on $F(x)=\int_0^x f(t)\,dt$.

R. P. Boas, Jr. (Evanston, Ill.).

Rooney, P. G. On the inversion of the Gauss transformation. Canad. J. Math. **9** (1957), 459–464.

The transform is

$$f(x) = (4\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty} e^{-\frac{1}{4}(x-t)^2}\phi(t)\,dt,$$

which is formally inverted by

$$\phi(x)=e^{-D^2}f(x)=\sum_{n=0}^{\infty}(-1)^n f^{(2n)}(x)/n!.$$

Sufficient conditions are given for convergence and for Abel summability of this series to $\frac{1}{2}\{\phi(x+)+\phi(x-)\}$; for convergence, $\phi \in L$ in a neighborhood of 0, ϕ is of bounded variation in a neighborhood of x, and

$$|t|^\lambda e^{-(x-t)^2/8}\phi(t) \in L(-\infty, \infty)$$

for some $\lambda>3$; for summability, $e^{-(x-t)^2/8}\phi(t) \in L(-\infty, \infty)$ and $\phi(x\pm)$ exists. As applications the author gives some formulas for Hermite polynomials, for example

$$H_n(\lambda x)=\sum_{r=0}^{[n/2]}\frac{n(n-1)\cdots(n-2r+1)}{r!}(\lambda^2-1)^r\lambda^{n-2r}H_{n-2r}(x).$$

R. P. Boas, Jr. (Evanston, Ill.).

Horváth, J. Singular integral operators and spherical harmonics. Trans. Amer. Math. Soc. **82** (1956), 52–63.

This paper consists of two parts. The first is concerned with singular integral operators of the convolution type such as the ones considered by A. Zygmund and the reviewer. It is shown that these operators are continuous in the spaces of distributions $\mathcal{D}'_{L^p}$, $1<p<\infty$. The second part is devoted to an algebraic formalism related to spherical harmonics. Let $A=\sum_1^\infty A_j$ be a graded commutative algebra and $e_1, \cdots, e_n$ a basis for A_1; assume that $e_1^2+\cdots+e_n^2=0$. Then if $x=x_1e_1+\cdots+x_ne_n$, $x^m \in A_m$ is a harmonic function of $x_1, \cdots, x_n$ for each m. If $e_1^2+e_2^2+\cdots+e_n^2=0$ is the only relation valid in the algebra then the components of x^m according to a basis for A_m form a complete set of linearly independent solid harmonics of degree m in $x_1, x_2, \cdots, x_n$. As an application the author considers singular integral operators H_j with kernels with values in A, namely with kernels $c_jx^j|x|^{-n-j}$, where c_j is a constant depending on j. The Fourier transforms of these kernels are equal to $(-i)^j x^j|x|^{-j}$. From this it follows that the H_j form a semigroup under composition, i.e. $H_nH_m=H_{n+m}$. *A. P. Calderón.*

du Plessis, Nicolaas. Some theorems about the Riesz fractional integral. Trans. Amer. Math. Soc. **80** (1955), 124–134.

This paper is concerned with properties of Riesz fractional integrals in Euclidean n-space E. The integral

of order α of f is given, except for a numerical factor, by

$$f_\alpha(P)=\int_E r^{\alpha-m}f(Q)dQ,$$

where r is the distance between P and Q. The results are the following: a) if $f\in \mathrm{Lip}\,\beta$, $0<\beta<1$, then $f_\alpha\in \mathrm{Lip}(\alpha+\beta)$, $0<\alpha+\beta<1$; b) if $f\in L^q$, $q>1$, $mq^{-1}<\alpha<mq^{-1}+1$, then $f_\alpha\in \mathrm{lip}(\alpha-mq)$; c) if $f\in L^q$, $0<\alpha<mq^{-1}$, then $f_\alpha\in L^r$, $r^{-1}=q^{-1}-\alpha m^{-1}$; d) if $f\in L^q$, $0<\alpha<m$, $2<q<\infty$, then $f_{\alpha/q}$ is finite outside a set which is of zero β-capacity for all $\beta>m-\alpha$; if $1\leqq q\leqq 2$, then $f_{\alpha/q}$ is finite outside a set of zero $(m-\alpha)$-capacity. The results in d) are best possible. On the other hand c), which was obtained before by S. Soboleff [C. R. (Dokl.) Acad. Sci. URSS (N.S.) **20** (1938), 5–9] and G. O. Thorin [Medd. Lunds Univ. Mat. Sem. **9** (1948), 1–58; MR **10**, 21], is given a particularly simple proof by reduction to the classical case $m=1$. *A. P. Calderón.*

du Plessis, Nicolaas. Spherical fractional integrals. Trans. Amer. Math. Soc. **84** (1957), 262–272.

The author defines a fractional integral for functions on a unit m-sphere and establishes theorems similar to those he proved for Riesz fractional integrals in the paper reviewed above. Briefly described, $f_\alpha(P)$ is the limit as $\rho\to 1$ of the Riemann-Liouville integral $f_\alpha(\rho, P)$ with origin 0 of the Poisson integral $f(r, P)$ of $f(P)$; it is not necessarily true that $f_{\alpha+\beta}(P)=(f_\alpha(P))_\beta$. The main theorems are that if $0<\alpha<\alpha+\beta<1$ and $f\in \mathrm{Lip}\,\alpha$ then $f_\beta\in \mathrm{Lip}(\alpha+\beta)$ [misprinted lip in the paper]; the theorem holds if Lip is replaced by lip in both places; if $f\in \mathrm{Lip}\,\alpha$, $q>1$, $1+m/q>\alpha>m/q$, then $f_\alpha\in \mathrm{lip}(\alpha-m/q)$; if $0<\alpha<m/q$ and $f\in L^q$ then $f_{\alpha\epsilon}\in L^r$ where $\alpha=m(1/q-1/r)$; if $f\in L^q$, $0<\alpha<m$, (a) if $2<q<\infty$, $f_{\alpha/q}$ is finite except in a set of zero β-capacity for $\beta>m-\alpha$, (b) if $1\leqq q\leqq 2$, $f_{\alpha/q}$ is finite except in a set of zreo $(m-\alpha)$-capacity. The following auxiliary theorem is of independent interest: $f(\rho, P)$, harmonic inside S, is the Poisson integral of an element of $\mathrm{Lip}\,\alpha$ if and only if $f(\rho', P)-f(\rho, P)=O((\rho'-\rho)^\alpha)$ uniformly in P; if O is replaced by o, Lip is replaced by lip.

R. P. Boas, Jr. (Evanston, Ill.).

See also: Ordinary Differential Equations: Wintner; Fage. **Partial Differential Equations:** Séminaire Schwartz; Šil'krut. **Probability:** Wintner; Girault; Ibragimov. **Statistics:** Kitagawa. **Elasticity, Plasticity:** Godfrey. **Structure of Matter:** Norman.

Ordinary Differential Equations

Wintner, Aurel. Ordinary differential equations and Laplace transforms. Amer. J. Math. **79** (1957), 265–294.

Let A denote the set of all functions f regular on the half-plane $\mathrm{Re}\,z>0$ and representable there as an absolutely convergent Laplace-Stieltjes transform $f(z)=\int_0^\infty e^{-zt}d\alpha(t)$ of some function α, defined on $0\leqq t<\infty$, which is of bounded variation on every finite subinterval and normalized by $\alpha(0)=0$, $\alpha(t+0)=\alpha(t)$. If $f\in A$, α is uniquely determined by f, and is denoted by α_f. The following theorem is proved: Let $f, g\in A$ and suppose $\int_0^t t^{-1}|d\alpha_g(t)|<\infty$, $\int_0^t t^{-2}|d\alpha_f(t)|<\infty$; then on $\mathrm{Re}\,z>0$ the general solution of $w''+g(z)w'+f(z)w=0$ has the form $w(z)=c_1zu(z)+c_2v(z)$, where $u, v\in A$. This result is analogous to the situation of a regular singular point at $z=0$. The non-linear case is also treated. Let $\{f_m\}$ $(m=0, 1, 2, \cdots)$ be a sequence of functions on $\mathrm{Re}\,z>0$ representable as $f_m(z)=\int_0^\infty e^{-zt}d\alpha_m(t)$, where α_m satisfies, for some $b>0$, $\sum_{m=0}^\infty[\alpha_m]b^m<\infty$, $\sum_{m=0}^\infty\{\alpha_m\}b^m<\infty$, with $[\alpha_m]=\int_0^\infty|d\alpha_m(t)|<\infty$, $\{\alpha_m\}=\int_0^t t^{-1}|d\alpha_m(t)|$. Let $f(z, w)=\sum_{m=0}^\infty f_m(z)w^m$ on $\mathrm{Re}\,z>0$, $|w|<b$. Suppose $f_0(z)\equiv 0$. Then there exist two positive numbers d^* and $b^*(\leqq b)$ having the following properties: If z_0, w_0 are complex numbers satisfying $\mathrm{Re}\,z_0>d^*$, $|w_0|<b^*$, then the solution $w=w(z; z_0, w_0)$ of the problem $dw/dz=f(z, w)$, $w(z_0)=w_0$, exists on $\mathrm{Re}\,z>d^*$ and is representable as a Laplace-Stieltjes integral $w(z)=\int_0^\infty e^{-zt}d\beta(t)$ which is absolutely convergent for $\mathrm{Re}\,z>d^*$. Further, $|w(z)|<b$ for $\mathrm{Re}\,z>d^*$. The assumption $f_0(z)\equiv 0$ is not needed until the last step in the proof, and this implies another result. An appendix collects together many of the known results, and adds certain new ones, concerning the non-local behaviour of solutions of $x'=f(t, x)$ on $0<t<\infty$. *E. A. Coddington.*

Olech, C. Sur certaines propriétés des intégrales de l'équation $y'=f(x, y)$, dont le second membre est doublement périodique. Ann. Polon. Math. **3** (1957), 189–199.

Consider $y'=g(x, y)$ with g defined and continuous for all x, y and $g(x, y)=g(x+p, y+q)$ for any integers p, q. The author proves the following theorems. (1) If $\varphi(x)$ $[\psi(x)]$ is a solution through (ξ, η) such that it is a maximum [minimum] solution for $x>\xi$ and a minimum [maximum] solution for $x<\xi$, then $x^{-1}\varphi(x)\to\beta$ $[x^{-1}\psi(x)\to\alpha]$ as $x\to+\infty$ or $-\infty$ where β $[\alpha]$ is a (finite) constant independent of (ξ, η). (2) For every $\lambda\in(\alpha, \beta)$ and every (ξ, η) there exist solutions $\omega(x)$, $\sigma(x)$ through (ξ, η) such that $x^{-1}\omega(x)\to\lambda$ as $x\to+\infty$ or $-\infty$ and $\lim x^{-1}\sigma(x)$, as $x\to+\infty$, does not exist: These results extend an analogous theorem of Poincaré [J. Math. Pures Appl. **1** (1885), 167–244], which requires uniqueness of every solution.

H. A. Antosiewicz (Providence, R.I.).

Opial, Z. Sur un système d'inégalités intégrales. Ann. Polon. Math. **3** (1957), 200–209.

Consider the system (1) $y_i'=f_i(x, y_1, \cdots, y_n)$ $(i=1, 2, \cdots, n)$ with f_i defined and continuous on some open set $\Omega\subset R\times R^n$. Let $P=(\xi, \eta_1, \cdots, \eta_n)\in\Omega$ and let $\varphi_i(x)$ $(i=1, 2, \cdots, n)$ be continuous functions which satisfy on some $I_\alpha=[\xi, \xi+\alpha]$, $\alpha>0$, the inequalities

$$(2)\quad \varphi_i(x)\leqq\eta_i+\int_\xi^x f_i[t, \varphi_1(t), \cdots, \varphi_n(t)]dt \quad (i=1, 2, \cdots, n).$$

First, the author proves this generalization of a theorem of Viswanatham [Proc. Indian Acad. Sci., Sect. A. **36** (1952), 335–341]: If each f_i is non-decreasing on Ω, i.e., if $(x, a_1, \cdots, a_n)$, $(x, b_1, \cdots, b_n)\in\Omega$ and $a_k\leqq b_k$ $(k=1, 2, \cdots, n)$ imply $f_i(x, a_1, \cdots, a_n)\leqq f_i(x, b_1, \cdots, b_n)$ for each i, then the functions φ_i satisfying (2) also satisfy $\varphi_i(x)\leqq\psi_i(x)$ $(i=1, 2, \cdots, n)$ on $I_\alpha\cap I_\beta$, where $\psi_1(x), \cdots, \psi_n(x)$ is a solution of (1) through P which is maximal on some $I_\beta=[\xi, \xi+\beta)$, $\beta>0$. The existence of such a solution $\psi_i(x)$ $(i=1, 2, \cdots, n)$ of (1) follows from a theorem of Ważewski [Ann. Soc. Polon. Math. **23** (1950), 112–166; MR **12**, 705]. Next, the author proves the following theorem. If for every $P\in\Omega$ and every set $\varphi_i(x)$ $(i=1, 2, \cdots, n)$ of continuous functions satisfying (2) on some I_α there exists a $\delta>0$ and a solution $y_i(x)$ $(i=1, 2, \cdots, n)$ of (1) through P such that $\varphi_i(x)\leqq y_i(x)$ on $[\xi, \xi+\delta]$, then each f_i is locally non-decreasing with respect to each y_k $(k\neq i)$ separately, and non-decreasing with respect to y_i in a sufficiently small neighborhood of every point in Ω. Finally, the author

gives a complete description of how a set $\varphi_i(x)$ $(i=1, 2, \cdots, n)$ of continuous functions can be constructed explicitly so as to satisfy (2) on some I_α.

H. A. Antosiewicz (Providence, R.I.).

★ **Ważewski, T. Sur une méthode topologique de l'examen de l'allure asymptotique des intégrales des équations différentielles.** Proceedings of the International Congress of Mathematicians, 1954, Amsterdam, vol. III, pp. 132–139. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam, 1956. $7.00.

Consider a system of first-order ordinary differential equations (*) $dx^i/dt=f_i(t, x_1, \cdots, x_n)$ $(i=1, \cdots, n)$ in which the f_i are continuous over an open set W of $(t, x_1, \cdots, x_n)$-space, and assume that through each point of W passes one and but one integral curve of (*). Let T be an open subset of W. The general problem discussed is, what integrals of (*) originating in T remain in T as t increases? Topological methods of the author and other Polish writers are described briefly and one basic theorem given in detail with several applications. This theorem is concerned only with T of such shape that once an integral curve has departed (with increasing t) from $\bar{T}$, the curve will not meet $\bar{T}$ again except possibly after the intervention of a finite t-interval. Assuming T to be of this shape, let S be the set of points of departure of integral curves from $\bar{T}$. The theorem alluded to states that, if $Z \subset T+S$, and if $Z \cap S$ is a retract of S but not a retract of Z, then an integral of (*) issuing from some point of $Z \cap T$ never will depart T (as t increase). (Let $A \subset B$; A is called a retract of B, if there is a continuous mapping of B into A leaving the points of A fixed.) As one of several applications given, it is shown that, in the case of a system of two equations $x'(t)=f(x, y, t)$, $y'(t)=g(x, y, t)$, there will be an integral curve, originating on an arbitrary plane $y=b$, $t=t_0$, for which $|x(t)|<u(t)$ and $|y(t)<v(t)$ for $t>t_0$, provided, for instance, $xf(t, x, y)>u(t)u'(t)$ for $|x|=u(t)$, $|y| \leqq v(t)$, $t>0$ and $yg(t, x, y)<v(t)v'(t)$ for $|x| \leqq u(t)$, $|y|=v(t)$, $t>0$. Analogous statements are made about the perturbed equations. *A. Douglis* (New York, N.Y.).

Thorne, R. C. The asymptotic solution of differential equations with a turning point and singularities. Proc. Cambridge Philos. Soc. **53** (1957), 382–398.

In view of the complexity of the results of this paper only a loose description is attempted here. In the differential equation $d^2w/dz^2=\{u^2p(z)+q(z)\}w$ let u be a large positive parameter and assume that $p(z)$ has a first order zero at $z=z_0$. Making appropriate assumptions on the behavior of $p(z)$ and $q(z)$, Olver [Philos. Trans. Soc. London. Ser. A. **247** (1954), 307–327; MR **16**, 695] has derived asymptotic series for the solutions of this equation that are uniformly valid in certain open unbounded domains D_z containing z_0. This is achieved by transforming the differential equation into one that asymptotically resembles Airy's equation. The description of D_z is given in terms of a new independent variable $z=f(\zeta)$. Using somewhat different transformations the author of the present paper extends Olver's results to differential equations whose coefficients possess, in addition to the zero at z_0, second order poles at a point $z=z_1$, and which may have regular, or even certain irregular, singularities at infinity. The domains D_z are described in terms of z itself. In general, D_z is an infinitely sheeted Riemann domain containing z_0 and z_1 as interior points. Also treated are equations in which $p(z)$ is symmetric with respect to z_1. In this case D_z includes the zero symmetric to z_0. The results yield, in particular, uniformly valid expansions of Bessel functions of large order and of Legendre functions of large order and degree. Further applications to Legendre functions are promised [see the paper reviewed below].

W. Wasow (Madison, Wis.).

Thorne, R. C. The asymptotic solution of linear second order differential equations in a domain containing a turning point and a regular singularity. Philos. Trans. Roy. Soc. London. Ser. A. **249** (1957), 585–596.

L'auteur étudie, pour les grandes valeurs positives du paramètre u, les développements asymptotiques des solutions de l'équation

$$(*) \qquad d^2w/dz^2=[u^2p(z)+q(z)]w,$$

où p et q sont des fonctions de la variable complexe z régulières sauf à $z=0$. Pour assurer l'univocité de la solution de (*), il est nécessaire d'introduire dans le domaine $D(z)$ décrit par z une coupure partant du point $z=0$ (qui est une singularité régulière de (*)).

Dans l'oeuvre analysé ci-dessus, l'auteur avait étudié le cas où les fonctions $p(z)$ et $q(z)$ étaient telles que

$$p(z)=z^{-2}(z_0-z)[a^2/z_0+z_0O(1)] \text{ et } q(z)=z^{-2}[b+zO(1)].$$

Le cas délicat étudié dans le présent travail est celui où la coupure est prise le long de l'axe réel positif, axe où l'on suppose justement que z_0 est située. (Les résultats précédents ne s'appliquaient pas dans cette hypothèse.)

Une transformation de l'équation, qui la met sous la forme $d^2y/dt^2=\{u^2(1+\alpha^2t^{-2})-\frac{1}{4}t^{-2}+g(t)\}y$ (avec $g(t)=z'^2z^{-2}q_1(z)+\frac{1}{4}t^{-2}-\frac{1}{2}\{z, t\}$, où $\{z, t\}$ désigne le schwarzien de z par rapport à t) permet de la comparer à l'équation plus simple obtenue en prenant $g(t)$ nul et dont deux solutions sont $t^{\frac{1}{2}}I_m(ut)$ et $t^{\frac{1}{2}}K_m(ut)$. (La paramètre α qui désigne m/u est supposé fixe.) On peut tirer de là pour les solutions de l'équation des développements asymptotiques dont les éléments sont des fonctions de Bessel et qui sont valables, uniformément par rapport à z, dans les domaines contenant les points $z=0$, $z=z_0 \pm i0$. (Les développements trouvés précédemment par l'auteur ne s'appliquaient pas à ces cas.) *R. Campbell* (Caen).

Zadiraka, K. V. On a system of non-linear differential equations, containing a small parameter in certain derivatives. Dokl. Akad. Nauk SSSR (N.S.) **109** (1956), 256–259. (Russian)

The author considers a system of differential equations depending on a parameter μ: $\dot{x}=f(x, z, t, t/\mu)$, $\mu\dot{z}=F(x, z, t)$, $x(t_0)=x_0$, $z(t_0)=z_0$, where $x=(x_1, \cdots, x_n)$, $z=(z_1, \cdots, z_m)$, and seeks the limiting form of the solution when $\mu \to 0$. He gives sufficient conditions that the solution approach a solution of a system $\dot{x}=f_0(x, z, t)$, $z=\phi(x, t)$, $x(t_0)=x_0$, where f_0 is the limit, as $T \to \infty$, of $T^{-1}\int_0^T f(x, z, t, v)dv$, and $z=\phi(x, t)$ satisfies $F(x, \phi, t) \equiv 0$. The reviewer found some details of the proof unclear.

W. Kaplan (Ann Arbor, Mich.).

Kostomarov, D. P. On the asymptotic behavior of solutions of systems of linear first order differential equations in the neighborhood of an irregular singular point. Dokl. Akad. Nauk SSSR (N.S.) **110** (1956), 918–921. (Russian)

The author has shown previously [same Dokl. (N.S.) **108** (1956), 1011–1013; MR **18**, 211] that the linear system $w_i'=\sum_{j=1}^n a_{ij}(z)w_j$ $(i=1, 2, \cdots, n)$ (in a neighborhood of infinity $a_{ij}(z)=\sum_{m=0}^{\infty} a_{ij}^{(m)}z^{k-m}$) has a fundamental formal

matrix solution. In this paper he studies the asymptotic relation between the formal solutions and the solutions. If $\overline{W}$ is a fundamental formal matrix solution, he shows that along each ray $\arg z=\varphi_0$ there is a fundamental matrix solution W with the property that $\overline{W}\simeq W$.

J. P. LaSalle (Notre Dame, Ind.).

Basov, V. P. Behavior of solutions of linear differential equations in the neighborhood of a singular point of irregular type. Mat. Sb. N.S. **40(82)** (1956), 339–380. (Russian)

In this paper the author continues his investigations of the behavior of solutions of $n\times n$ matrix equations $\dot{X}=(P^0+P(t))X$ near an irregular singular point [Dokl. Akad. Nauk SSSR (N.S.) **80** (1951), 301–304; Akad. Nauk SSSR. Prikl. Mat. Meh. **18** (1954), 313–328; Ukrain. Mat. Ž. **8** (1956), 97–109; see also, e.g., I. M. Rapoport, Ukrain. Mat. Ž. **8** (1956), 110–111; V. V. Horošilov, Leningrad. Gos. Univ. Uč. Zap. **137**. Ser. Mat. Nauk **19** (1950), 180–197; MR **13**, 557; **16**, 360; **18**, 40]. His present assumptions are: (1) P^0 is a constant matrix (with no restrictions upon its canonical form); (2) $P(t)$ is given by a series, uniformly convergent on $t\geqq T>0$, of the form

$$P(t)=\sum_{l_1+\cdots+l_k>0} t^{-(l_1\gamma_1+\cdots+l_k\gamma_k)}P^{(l_1,\cdots,l_k)}$$

where k is any (fixed) positive integer, $\gamma_1, \cdots, \gamma_k$ are in commensurable real positive constants with γ_1 being the reciprocal of an integer, and $P^{(l_1,\cdots,l_k)}$ are matrices that are either constant or continuous periodic functions of t on $t\geqq T$ with a common period. The main result concerns an explicit construction of a fundamental system of solutions in the form of infinite series which are uniformly convergent on $t\geqq T$.

The theorem on which this construction is based may be stated as follows. Let

$$X_i=t^{\beta_i}P_{ii}X_i+Q_i(t, X_1, \cdots, X_l) \quad (i=1, 2, \cdots, l_1),$$

$$X_i=Q_i(t, X_1, \cdots, X_l) \quad (i=l_1+1, \cdots, l),$$

$$Q_i(t, X_1, \cdots, X_l)=\sum_{j=1}^{l_1} t^{\beta_i}Q_{ij}(t)x_j+\sum_{j=l_1+1}^{l} t^{-\alpha_i}R_{ij}(t)X_j,$$

where P_{ii} is a constant $m_i\times m_i$ matrix whose eigenvalues have non-zero real parts, $Q_{ij}(t)$, $R_{ij}(t)$ are $m_i\times m_j$ matrices, continuous and bounded on $t\geqq T$, $Q_{ij}(t)\to 0$ as $t\to+\infty$, and α_i, β_i are (real) constants satisfying $\alpha_i>1$, $\beta_i>-1$. Then to every α_j there corresponds an m_j-columned solution

$$X_i(t)=t^{-(\alpha_j+\beta_i)}U_i(t) \quad (i=1, 2, \cdots, l_1),$$

$$X_i(t)=\delta_{ij}I+t^{-(\alpha_j-1)}U_i(t) \quad (i=l_1+1, \cdots, l),$$

where $U_i(t)$ are $m_i\times m_j$ matrices bounded on $t\geqq T$. The proof follows a general method of Erugin [Trudy Mat. Inst. Steklov **13** (1946); MR **9**, 509].

H. A. Antosiewicz (Providence, R.I.).

Borghi, D. C. Soluzioni generali delle equazioni differenziali lineari omogenee del secondo ordine. Ann. Geofis. **8** (1955), 151–165.

Dolaptschiew, Blagowest; und Tschobanow, Iwan. Über eine Differentialgleichung von J. Halm. Math. Nachr. **15** (1956), 197–200.

The authors show that the equation

$$4(ax^2+bx+c)^2z''+[(ax^2+bx+c)\theta+\tfrac{1}{2}(b^2-4ac)]z=0$$

has the solution

$$z=(C_1\cos\lambda\xi+C_2\sin\lambda\xi)(ax^2+bx+c)^{\frac{1}{2}},\ \theta=a+4\lambda^2>a,$$

$$z=(C_1\cosh\lambda\xi+C_2\sinh\lambda\xi)(ax^2+bx+c)^{\frac{1}{2}},\ \theta=a-4\lambda^2<a,$$

where $\xi=\int(ax^2+bx+c)^{-1}dx$; also that the equation

$$4(ax^2+bx+c)^2z''+[-4a\alpha(1+\alpha)(ax^2+bx+c)+(1-\alpha^2)(b^2-4ac)]z=0$$

has the solution

$$z=[C_1\int(ax^2+bx+c)^{-\alpha-1}dx+C_2](ax^2+bx+c)^{(\alpha+1)/2}.$$

For particular values of a, b, c these equations give cases of an equation studied by Halm [Trans. Roy. Soc. Edinburgh **41** (1906), 651–676], viz.

$$(x^2+1)^2z''+(Ax^2+B)z=0.$$

D. C. Gilles (Manchester).

Giuliano, Landolino. Sopra le equazioni differenziali lineari autoaggiunte del terzo ordine. Boll. Un. Mat. Ital. (3) **12** (1957), 16–18.

Si dimostra che la classe delle equazioni differenziali lineari omogenee del terzo ordine il cui integrale generale è dato da un'espressione quadratica $z=ay_1^2+2by_1y_2+cy_2^2$, dove $y_1(x)$ e $y_2(x)$ sono due integrali linearmente indipendenti dell'equazione differenziale $y''=q_1(x)y'+q_2(x)y$ e a, b, c costanti arbitrarie, è costituita unicamente dalle equazioni autoaggiunte. *Riassunto dell'autore.*

Vasilache, Sergiu. Sur une nouvelle méthode de résolution des équations différentielles linéaires. Com. Acad. R. P. Române **2** (1952), 409–412. (Romanian. Russian and French summaries)

Taucer, Grazia. Sulla soluzione di particolari equazioni lineari. Matematiche, Catania **10** (1955), 20–25.

Fage, M. K. Construction of operators of transformation and solution of a problem of moments for ordinary linear differential equations of arbitrary order. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 1(73), 240–245. (Russian)

Let (*) $l(y)\equiv y^{(n)}+p_{n-1}(x)y^{(n-1)}+\cdots+p_0(x)y=f(x)$ be defined in Δ: $0\leqq x<l$, $l\leqq\infty$, where $p_k\in C^k$ ($k=0, \cdots, n-1$). The problem is to find integral transforms which relate the general equations (*) to the simplest case where $l(y)=l_0(y)=y^{(n)}$; for $n=2$ this problem was considered from two different points of view by the author [Dokl. Akad. Nauk SSSR (N.S.) **99** (1954), 909–912; MR **17**, 39] and A. Ya. Povzner [Mat. Sb. (N.S.) **23(65)** (1948), 3–52; MR **10**, 299]. These previous results are generalized for $n>2$.

J. L. Massera (Montevideo).

Delsarte, Jean; et Lions, Jacques Louis. Transmutations d'opérateurs différentiels dans le domaine complexe. C. R. Acad. Sci. Paris **244** (1957), 832–834.

Let H be the linear space of entire functions. Let A be a differential operator of order m of the form

$$A=D^m+a_1(x)D^{m-1}+\cdots+a_m(x),$$

where the coefficients are entire. Theorem: There exists a linear operator T on H such that $D^mT=TA$; moreover, T is an isomorphism of H onto itself. Thus, any two differential operators of the same degree on H are conjugates. T is given explicitely as an infinite order differential operator; it is unique, up to a factor U which may be any operator in the group of isomorphisms of H which com-

mute with D^m. Numerous applications are outlined; the authors also state that the result does not extend to a similar conjugacy theorem for differential operators of class C^∞. *R. C. Buck* (Madison, Wis.).

Gumowski, Igor. Un critère de stabilité sous forme d'une équation intégrale. C. R. Acad. Sci. Paris **244** (1957), 2004–2007.

The author considers equations of the form $L(D)y(t)=f(t)$, where $L(D)$ is a linear differential operator with constant coefficients, of finite or infinite order; he regards such an equation as defining an electrical circuit. He gives reasons for supposing that such a circuit is stable in a certain sense if and only if

$$\int_{-\infty}^{\infty} e^{j\omega t} L^{-1}(j\omega)d\omega=0 \quad (t<0),$$

and illustrates his argument by giving examples of operators $L(D)$ that satisfy this condition and others that do not. *F. Smithies* (Cambridge, England).

Brodskiĭ, M. S. The inverse problem for systems of linear differential equations containing a parameter. Dokl. Akad. Nauk SSSR (N.S.) **112** (1957), 800–803. (Russian)

The paper is devoted to the following problem. Let

$$(*) \qquad \frac{dy_i}{dx}=\frac{i}{\lambda}\sum_{j=1}^{n} b_{ij}(x)y_j \quad (1\leq i\leq n,\ 0\leq x\leq l)$$

have $W(x, \lambda)$ as fundamental matrix solution, i.e., $dW(x, \lambda)/dx=(i/\lambda)b(x)W(x, \lambda)$, where $b(x)$ is the matrix $[b_{ij}(x)]$ and $W(0, \lambda)=I$, the unit matrix. When is $b(x)$ completely determined by $W(\lambda)=W(l, \lambda)$?

The matrix function $W(\lambda)$ is said to belong to the class M^+ if (i) $W(1/z)$ is an entire function of $z=\lambda^{-1}$, (ii) $W(\lambda)\to I$ $(\lambda\to\infty)$, (iii) $W(\lambda)W^*(\lambda)=I$ for all real λ, (iv) $W(\lambda)W^*(\lambda)\geq I$ for all λ in the upper half-plane. Any such function has an expansion of the form $W(\lambda)=I+i\lambda^{-1}H+\cdots$ in negative powers of λ, where $H\geq 0$; the trace $\mathrm{tr}(H)$ of H is called the weight of $W(\lambda)$. If one can write $W(\lambda)=W_2(\lambda)W_1(\lambda)$, where W_1 and W_2 belong to M^+, $W_1(\lambda)$ is called a divisor of $W(\lambda)$.

The bounded linear operator A in a Hilbert space $\mathfrak{H}$ is said to belong to the class K^+ if (i) the spectrum of A is $\{0\}$ only, (ii) $E=i(A^*-A)$ is of finite rank, and all its non-zero eigenvalues are positive, (iii) if (e_α) is a full orthonormal set of non-trivial eigenfunctions of E, the vectors $A^n e_\alpha$ form a fundamental set in $\mathfrak{H}$.

The author proves that $W(\lambda)$ is in M^+ if and only if it is a characteristic matrix-function of an operator A of the class K^+, and that in these circumstances $W_1(\lambda)$ is a divisor of $W(\lambda)$ if and only if it is the projection of $W(\lambda)$ on some invariant subspace of A; the terms used here are defined in an earlier paper [same Dokl. (N.S.) **97** (1954), 761–764; MR **16**, 836]. A sufficient condition is given for $W(\lambda)$ to have exactly one divisor of each weight $l_1<l$, where l is the weight of $W(\lambda)$.

These results are now applied to the system of equations (*). The system is said to be normalized if $\mathrm{tr}[b(x)]=1$ for all x; a general system can be normalized by a change of independent variable if $\mathrm{tr}[b(x)]\neq 0$ for almost all x. Suppose now that two systems in the same interval $[0, l]$ with Hermitian non-negative matrices $b^{(1)}(x)$ and $b^{(2)}(x)$ are both normalized, that the functions $b_{ij}^{(1)}(x)$ have absolutely continuous first derivatives, and that the rank of $b^{(1)}(x)$ is 1 for all x; then the equality $W^{(1)}(\lambda)=W^{(2)}(\lambda)$ implies that $b^{(1)}(x)=b^{(2)}(x)$ for all x. The same implication holds if $b^{(2)}(x)$ is Hermitian non-negative definite, its elements being summable functions of x, and $b^{(1)}(x)=\xi^*(x)\xi(x)$, where $\xi(x)$ is a row vector that is sectionally constant as a function of x, and two successive constant values of $\xi(x)$ are never orthogonal to one another. *F. Smithies* (Cambridge, England).

Röhrl, Helmut. Das Riemann-Hilbertsche Problem der Theorie der linearen Differentialgleichungen. Math. Ann. **133** (1957), 1–25.

Consider the system of formal differential equations (1) $dy^i/dx=R^i{}_j(x)y^j(x)$ $(i, j=1, \cdots, n)$, and let $X'=\{x_i|i=1, \cdots, k\}$. Let l be a loop based on x_0 and write $[l]$ for the corresponding element of $\pi(X-X', x_0)$. Let X' and the homomorphism h: $\pi(X-X', x_0)\to \mathrm{GL}(n, C)$ be assigned. The Riemann-Hilbert problem for X the complex sphere is that of establishing existence of rational functions $R^i{}_j(x)$, with X' the singular points of the n independent vectors $\{Y_m=y_m{}^1, \cdots, y_m{}^n\}$, such that on passing around l there results (2) $Y_m{}'=A_m{}^r(x)Y_r$, where $A_m{}^r(x)\in \mathrm{GL}(n, C)$ is the matrix $h([l])$. This classical R. H. problem is settled. The writer successfully treats the case that X is a compact or an open Riemann surface. He uses the universal covering surface for his formulation of the regularity requirements of the problem. Thus (2) is interpreted on this surface, and $(R^i{}_j)$ is to be a non-singular matrix of meromorphic functions on this surface, while $\{Y_m\}$ denote solution vectors which are to be schlicht functions there. In contradistinction to the classical function theory treatments the methods used here are largely those of fibre spaces. These are natural tools since the problem is one of extending a local to a global solution and so is ultimately one on the existence of sections. The fibre spaces are introduced in terms of plausible coordinate functions, $g_{ij}(x)$. For instance, with $\{U_i\}$ a fine enough open cover, $g_{ij}(x)=h[\lambda_i l_{ij}(x)(l_{ji}(x))^{-1}\lambda_j^{-1}]$, where w_i is a point chosen in U_i, and λ_i and $l_{ij}(x)$ are the paths from x_0 to w_i and from w_i to $x\in U_j$. Other fibre spaces play a role also. The paper includes some results on complex analytic fibre spaces over Riemann surfaces with $\mathrm{GL}(n, C)$, or a complex Lie group, as structure group. *D. G. Bourgin* (Paris).

Fadle, J. Eigenwertprobleme von Affinoren und ihre Anwendung zur Lösung von zwei vektorischen Differentialgleichungen. Z. Angew. Math. Mech. **36** (1956), 248–250.

Bellman, Richard. On the linear differential equations whose solutions are the products of solutions of two given linear differential equations. Boll. Un. Mat. Ital. (3) **12** (1957), 12–15.

The purpose of this paper is to illustrate the application of a result in matrix theory to the problem of determining the linear differential equation of which the solutions are the products of the solutions of two given linear differential equations. The author shows that if $A(t)$ and Y are $m\times m$ matrices, $B(t)$ and Z are $n\times n$ matrices, and C and X are $m\times n$ matrices such that $dY/dt=A(t)Y$, $Y(0)=I$, $dZ/dt=ZB(t)$, $Z(0)=I$, then the solution of $dX/dt=A(t)X+XB(t)$, $X(0)=C$ is $X=YCZ$. Further, he outlines a technique for obtaining the linear differential equation of order mn of which the solutions are the products of the solutions of two linear differential equations, one of order m and one of order n. The differential equation obtained is analogous to Watson's fourth order equation, the solutions of which are pairs of confluent hypergeometric functions. *L. J. Slater*.

Diliberto, Stephen P. A note on linear ordinary differential equations. Proc. Amer. Math. Soc. 8 (1957), 462–464.

Reid, William T. Remarks on a matrix transformation for linear differential equations. Proc. Amer. Math. Soc. 8 (1957), 708–712.

Let x be an $n\times 1$ matrix, and let $A(t)$ be $n\times n$; let A^i be the ith column, and A_j the jth row, of A; let A^T be the transpose of A, and A^* the conjugate transpose; and let $x'=dx/dt$. If $x=Py$, where x is a solution of (*) $x'=Ax$, then y satisfies $y'=By$, where $B=P^{-1}AP-P^{-1}P'$.

Diliberto has shown for real-valued A [Contributions to the theory of nonlinear oscillations, Princeton, 1950, pp. 1–38; MR 11, 665] that, if the columns of P are obtained by the Gram-Schmidt orthogonalization process from an $n\times n$ matrix whose columns are linearly independent solutions of (*), then B is upper-triangular. He also has stated that the b_{ij} are bounded if the a_{ij} are; in the present paper he completes the proof of this statement by deriving the formulas $b_{ii}=P_i^TAP^i$, $b_{ij}=P_i^T(A+A^T)P^j$ $(i<j)$.

Reid gives neat matrix proofs of the above results for complex-valued functions, with the additional result that B has real diagonal elements. He does this by showing that B is expressible as a product of upper-triangular matrices with real diagonal elements, and by deriving the formula $B^*+B=P^*(A^*+A)P$. A few additional remarks are made, and it is observed that these results were obtained, though not stated as such, by Perron [Math. Z. 32 (1930), 465–473], and so should be credited to him.

W. J. Coles (Salt Lake City, Utah).

Rehlickiĭ, Z. I. On stability of solutions of certain linear differential equations with retarded argument in a Banach space. Dokl. Akad. Nauk SSSR (N.S.) **111** (1956), 29–32. (Russian)

The author considers the equation $dy/dt-A(t)y(t-a)=x(t)$, $0\leq t<\infty$, $a>0$, $y(t)=\phi(t)$ for $t\leq 0$, where $A(t)$ is an operator. Under appropriate conditions concerning $A(t)$, the behavior of the solution of this equation as $t\to\infty$ is determined by the behavior of solutions of the equation $dy/dt-\lambda y(t-a)=0$, with the same initial conditions, where λ is in the spectrum of $A(t)$. *R. Bellman.*

Gavrilov, N. I. On Lyapunov stability in the presence of zero eigenvalues. Mat. Sb. N.S. **41(83)** (1957), 7–22. (Russian)

Complete proofs of theorems announced previously [Dokl. Akad. Nauk SSSR (N.S.) **84** (1952), 425–428, 657–660; MR **14**, 275, 276]. *J. L. Massera* (Montevideo).

Barbălat, I. Remarks on certain theorems of Malkin on the existence of the function of Lyapunov. Gaz. Mat. Fiz. Ser. A. **8** (1956), 617–623. (Romanian)

This paper contains a theorem on the existence of Lyapunov functions in the case of uniform-asymptotic stability which was proved by the author in 1955. As the author himself acknowledges his result is less general than those of Kurzweil [Czechoslovak Math. J. **6(81)** (1956), 217–259, 455–484; MR **19**, 33] and the reviewer [Ann. of Math. (2) **64** (1956), 182–206; MR **18**, 42].

J. L. Massera (Montevideo).

★ **Зубов, В. И. [Zubov, V. I.] Методы А. М. Ляпунова и их применение. [The methods of A. M. Lyapunov and their application.]** Izdat. Leningrad. Univ., Moscow, 1957. 241 pp. 12.35 rubles.

This book is one more of the long and superb series of recent Soviet books on differential equations. More than ever it emphasizes the supremacy of Soviet Science in this particular chapter of mathematics. The shadow of Lyapunov hovers over it all. The book is therefore appropriately dedicated to the centenary of the birth of Lyapunov and begins with a portrait of that outstanding savant.

Briefly speaking, this work deals with the stability of dynamical systems à la Lyapunov in the most general sense possible: stability of singular points of standard systems, of their invariant closed sets, of solutions of partial differential equations. The basic method is Lyapunov's second, whose converse (sufficiency) is constantly and profoundly dealt with in accordance with the latest results (notably of Krasovskiĭ). In addition the first method of Lyapunov is also studied at length and with notable extensions. By way of comparison this first rate book is not a text but has rather the character of the general volumes of the Colloquium Series. A detailed analysis follows.

Ch. I. Stability of invariant sets of dynamical systems in metric spaces. After preliminary general definitions general dynamical systems in a metric space are defined following Niemitzki-Stepanov, as a field of operations of a one-parameter additive group. Stability, ordinary or asymptotic, uniform or otherwise, are all defined for closed invariant sets. Theorems stating necessary and sufficient conditions are given. For asymptotic stability they include Krasovskiĭ's condition: non-existence of any complete trajectory in some neighborhood of the set. They are also expressed in terms of a functional V which plays the role of the Lyapunov function in this very general context. Application is made to distance estimates of a motion from an invariant set. For asymptotic stability the maximal region for such stability is dealt with, and in the Euclidean space case, conditions assuring that it be the whole space are also given.

Ch. II. Application to ordinary differential systems. Extensive application is made of the general theory to autonomous and non-autonomous systems (isolated singularity), and in particular to analytical and to homogeneous systems.

Ch. III. The first method of Lyapunov. This method, valid only for analytical systems, rests upon the explicit series expansions of the solutions near a singularity. It is extensively dealt with in the chapter.

Ch. IV. Stability of invariant sets of general systems. Take an n-vector system $\dot{x}=X(x;t)$, $t\geq 0$ which in $E_n\times[t\geq 0]$ satisfies the condition for the existence of solutions, but not necessarily conditions for uniqueness. Given (x^0,t_0), let $F_{t_0}{}^t(x^0,t_0)$ denote the set of all points for $t\geq t_0$ on the solutions through (x^0,t_0) for $t\geq t_0$. The discourse applies to the collection $\{F\}$ (non-metrized) called a general system and the theory developed in Ch. I, is applied to them (the definitions are repeated for this case).

Ch. V. Application to partial differential equations. This is assuredly the most striking part of the book. Take a system $u_t=f(u,u_x,x)$ (finite-dimensional vectors throughout) and a vector $\phi(x)$ with Φ as its metrized space (dim ϕ=dim u). Suppose everything adequately continuous and that for every ϕ there is a solution $u(\phi,t)$ such that $u(\phi,0)=\phi$. Thus $u(\phi,t)$ is a dynamical system in Φ to which most of the results of Ch. I are extended.

S. Lefschetz (Princeton, N.J.).

Fogagnolo-Massaglia, Bruna. Sulle vibrazioni quasi-armoniche di un sistema dissipativo con elasticità costante a tratti. Atti Sem. Mat. Fis. Univ. Modena. **7** (1953–54), 167–181 (1956).

It is stated at the outset that a known sufficient condition that $\lim_{t\to\infty} q(t)=0$, where $q(t)$ satisfies $\ddot{q}+2c\dot{q}+\omega^2(t)q=0$ with $\omega(t)$ periodic, is that the constant c be sufficiently large. This is incorrect, since, when $\omega(t)\equiv 0$, $q=$constant is always a solution no matter how large c is. The theorem is, however, certainly true if modified so as to require ω^2 to have a positive lower bound. The author is concerned with the estimation of the least value of c_0 which will necessitate $q(t)\to 0$ for all $c>c_0$ in the special case that ω is a step function assuming two constant positive values over alternate half periods. He also investigates an upper limit for the period having the same effect. *D. C. Lewis, Jr.* (Baltimore, Md.).

Braunbek, Werner. Erzwungene Schwingungen eines einfachen nichtlinearen Systems. I. Die Differentialgleichungen und ihre stationären Lösungen. Z. Physik **147** (1957), 297–306.

Braunbek, Werner; und Sauter, Elmar. Erzwungene Schwingungen eines einfachen nichtlinearen Systems. II. Die nichtstationären Bewegungen. Z. Physik **147** (1957), 507–519.

The authors discuss the standard second order differential equation with constant coefficients, a sinusoidal right member, and a non-linear restoring force given by $kx+x^3$. The familiar method of slowly varying amplitude and phase is used to discuss steady state and transient behaviour with and without linear damping. — Insufficient references are given to the extensive literature. *N. Levinson* (Cambridge, Mass.).

Hale, Jack K. On a class of linear differential equations with periodic coefficients. Illinois J. Math. **1** (1957), 98–104.

The present paper concerns systems of linear differential equations of the form

$$(*) \qquad y''+A(\lambda)y=\lambda\phi(t;\lambda)y+\lambda\psi(t;\lambda)y',$$

where λ is a real parameter, $y=(y_1,\cdots,y_n)$, $A(\lambda)=\mathrm{diag}(\sigma_1^2,\cdots,\sigma_n^2)$, ϕ, ψ are $n\times n$ real matrices whose elements are real periodic functions of t of period $T=2\pi/\omega$, are L-integrable in $[0,T]$, are analytic in λ, and have mean value zero. It is supposed that each $\sigma_j^2(\lambda)$ $(j=1,\cdots,n)$ is a real positive analytic function of λ with $\sigma_j(0)\not\equiv\sigma_h(0) \pmod{\omega i}$ $(j\neq h;\ j,h=1,\cdots,n)$. It is further supposed that $\phi=(\phi_{11},\phi_{12};\phi_{21},\phi_{22})$, $\psi=(\psi_{11},\psi_{12};\psi_{21},\psi_{22})$, where ϕ_{11}, ψ_{11} are $\mu\times\mu$ matrices, ϕ_{22}, ψ_{22} are $\nu\times\nu$ matrices, and $\mu+\nu=n$, $0\leq\mu\leq n$, $0\leq\nu\leq n$. The following theorem is proved: (**) If (α) ϕ_{11}, ϕ_{22}, ψ_{21}, ψ_{12} are even in t, (β) ϕ_{21}, ϕ_{12}, ψ_{11}, ψ_{22} are odd in t, then for $|\lambda|$ sufficiently small all solutions of (*) are bounded in $(-\infty,+\infty)$.

A method successively developed by L. Cesari, R. A. Gambill, and J. K. Hale is used for the proof. This method, based on a convergent process of successive approximations, has been consistently applied to the proof of sufficient conditions for boundedness of the solutions of linear systems with periodic coefficients [Cesari, Atti Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. **11** (1941), 633–695; MR **8**, 208; Hale, Riv. Mat. Univ. Parma **5** (1954), 137–167; MR **17**, 36; Gambill, ibid. **5** (1954), 169–181; MR **17**, 36]; the proof of criteria for the existence of unbounded solutions of the same systems [Gambill, ibid **6** (1955), 37–43; MR **17**, 849]; the proof of general existence theorems for periodic solutions of weakly nonlinear autonomous a nalytic systems [Hale, ibid. **5** (1954), 281–311; MR **17**, 1088]; the proof of existence theorems for harmonic, subharmonic, ultraharmonic solutions of weakly nonlinear periodic analytic systems [Gambill and Hale, J. Rational Mech. Anal. **5** (1956), 353–394; MR **17**, 1086; Cesari and Hale, Proc. Amer. Math. Soc. **8** (1957), 757–764; MR **19**, 142]. Statement (**) contains for $\mu=n$ a previous result proved by the same method by Gambill [MR **17**, 36]. The conclusion of (**) does not necessarily hold if conditions α, β are not met for some μ, ν, as has been proved by examples [Cesari, MR **8**, 208; Gambill, MR **17**, 849] and by general criteria [Gambill, MR **17**, 849]. *L. Cesari* (Lafayette, Ind.).

Blehman, I. I. On the stability of periodic solutions of quasi-linear autonomous systems with several degrees of freedom. Dokl. Akad. Nauk SSSR (N.S.) **112** (1957), 183–186. (Russian)

An autonomous system $y'=Ay+\mu f(y,\mu)$ is considered, where y, f are n-vectors, A is an n by n matrix, and μ is a parameter. It is assumed that the characteristic roots of A have simple elementary divisors, but otherwise there may be multiple roots, and an arbitrary number of pure imaginary roots of any multiplicity. The real parts of all roots are assumed to be non-positive. (The author says non-negative, but his formula (2) contradicts this.) Upon making a linear transformation of the y-space and a change in the time scale, this system can be reduced to one of the form

$$\dot{x}_s=(1-\mu\delta_1)[\lambda_s x_s+\mu F_s(x_1,\cdots,x_s,\mu)] \quad (s=1,\cdots,n),$$

where the λ_s are the characteristic roots of A. It is assumed that at least one λ_s has the form $\lambda_s=in_s$, n_s an integer, not zero. Also F_s is assumed analytic in $x_1,\cdots,x_s$ and for small $\mu\geqq 0$. For $\mu=0$ there is a solution of this system of period 2π, and if certain further conditions are satisfied [see, e.g., Coddington and Levinson, Contributions to the theory of nonlinear oscillations, vol. 2, Princeton, 1952, pp. 19–35; MR **14**, 891] there will exist periodic solutions of period 2π for small μ reducing to the given one for $\mu=0$. The author gives a sufficient condition for these solutions to be asymptotically stable (the real parts of all roots of some algebraic equations are negative), and a sufficient condition for instability (at least one root has a positive real part). *E. A. Coddington.*

Volosov, V. M. Periodical solutions of a non-linear equation of auto-oscillations. Dokl. Akad. Nauk SSSR (N.S.) **115** (1957), 20–22. (Russian)

This is a study of the periodic solutions of

$$(1) \qquad \ddot{x}+Q(x)=\varepsilon f(x,\dot{x},\varepsilon),$$

$$\mathrm{sign}\ Q(x)=\mathrm{sign}\ x,\quad \int_0^{\pm\infty} Q(x)dx=\infty,$$

for ε small; the system

$$(2) \qquad \ddot{x}+Q(x)=0$$

has then only periodic solutions. The initial conditions are taken as

$$x(0)=F_1(\varepsilon)=\sum_{n=0}^{\infty} F_{1n}\varepsilon^n,\quad \dot{x}(0)=0,\quad F_{10}>0.$$

The solution passes then very close to the periodic solution of (2) with initial value F_{10} and oscillates between $F_1(\varepsilon)$ and a certain $F_2(\varepsilon)=\sum_{n=0}^{\infty} F_{2n}(\varepsilon^n)$, with a period $T(\varepsilon)=\sum_{n=0}^{\infty} T_n(\varepsilon^n)$.

Two expressions $A_1(F_{10}, F_{20})$ and $A_2(F_{10}, F_{20})$ are given whose vanishing guarantees that (F_{10}, F_{20}) are generating values of solutions of (2) for periodic solutions of (1). The other coefficients F_{1n} are calculated in the obvious manner by substitution in (1) and identification of powers of ε. *S. Lefschetz* (Princeton, N.J.).

Crum, M. M. On the Sturm-Liouville expansion. Quart. J. Math., Oxford Ser. (2) **6** (1955), 288–292.

Nel classico teorema di equiconvergenza di Haar, relativo alle autosoluzioni normalizzate del sistema $\psi''+\{\lambda-q(x)\}\psi=0, \psi'(0)=h\psi(0), \psi'(\pi)=h\psi(\pi)$, l'ipotesi che $q(x)$ sia a variazione limitata in $(0, \pi)$ può esser sostituita da quella che $q(x)$ sia di classe $L(0, \pi)$.

G. Scorza Dragoni (Zbl. **66** (1957), 336).

Hille, Einar. On a class of orthonormal functions. Rend. Sem. Mat. Univ. Padova **25** (1956), 214–249.

Given a positive continuous function $b(x)$ such that $xb(x)^{-1} \in L(-\infty, \infty)$, consider the boundary value problem $b(x)y''-\lambda y=0$, $y(x)=y(x, \lambda) \in C[-\infty, \infty]$. The problem has a pure discrete point spectrum $\{\lambda_n\}$ with corresponding eigenfunctions $\{\omega_n(x)\}$ such that $\lambda_0=0$, $\lambda_n<0$ for $n>0$, $\sum_n |\lambda_n|^{-1}<\infty$, $\int_{-\infty}^{\infty}\omega_j(x)\omega_k(x)b(x)^{-1}dx=\delta_{jk}$ [Hille, J. Analyse Math. 3 (1954), 81–196, Chap. 4; MR **16**, 45]. The present paper is devoted to a detailed study of the asymptotic properties of the system $\{\lambda_n, \omega_n(x)\}$ in relation to the asymptotic properties of $b(x)$. Thus it is proved that the exponent of convergence of $\sum_n |\lambda_n|^{-\sigma}$ is $\frac{1}{2}$ if $b(x)^{-1} \in L(-\infty, \infty)$, and it lies in the closed interval $[\frac{1}{2}, 1]$ when this condition does not hold, the upper limit 1 being approached when the rate of growth of $b(x)$ becomes slower and slower. It is also proved that $M_n=\sup_x |\omega_n(x)|=o(|\lambda_n|)$ if $x^2b(x)^{-1} \in L(-\infty, \infty)$ and, if $b(x)$ is of slow rate of growth, then M_n grows faster than any power of $|\lambda_n|$. These results are obtained by a series of lemmas, yielding a study of the zeros of the entire function $D(\lambda)$ which is the Wronskian of two suitably chosen "subdominant" solutions of the equation. In the final section § 6, the results are extended to the case of the diffusion equation $b(x)y''+a(x)y'-\lambda y=0$, $y(x, \lambda) \in C[\alpha, \beta]$, under the assumption that the endpoints α and β are entrance boundaries in the sense of W. Feller.

K. Yosida (Tokyo).

Štraus, A. V. On eigenfunction expansion of a second order boundary problem of a semi-axis. Izv. Akad. Nauk SSSR. Ser. Mat. **20** (1956), 783–792. (Russian)

Let p, q be real-valued measurable functions on $0\leq x<\infty$, and suppose that for any $b>0$, $\int_0^b |p(x)|^{-1}dx<\infty$, $\int_0^b |q(x)|dx<\infty$. Let $Ly=-(py')'+qy$, and assume that L is of the limit point type at ∞. The boundary value problem $Ly=ly$, $y(0)=\theta(l)(py')(0)$, is considered on $0\leq x<\infty$. Here θ is any meromorphic function having a non-negative imaginary part in the upper half-plane and real on the real axis. The case $\theta(l)\equiv\infty$ is also admitted, and in case $\theta(l)=\infty$ the boundary condition is interpreted as $(py')(0)=0$. Let L_0 denote the minimal closed symmetric operator in $L^2(0, \infty)$ associated with L, and let $L_{\theta(l)}$ denote the quasi-self-adjoint extension of L_0 determined by $\theta(l)$. This operator $L_{\theta(l)}$ satisfies $L_0 \subset L_{\theta(l)} \subset L_0^*$ and its domain consists of those elements in the domain of L_0^* satisfying the boundary condition $y(0)=\theta(l)(py')(0)$. There exists a generalized resolution of the identity F associated with $L_{\theta(l)}$ via $(L_{\theta(l)}-l)^{-1}=\int_{-\infty}^{\infty}(\lambda-l)^{-1}dF(\lambda)$, $\operatorname{Im} l\neq 0$. The author shows how the expansion theorem and Parseval equality associated with F can be expressed in terms of a single spectral function ρ if one expands in terms of an appropriate solution of $Ly=ly$ which is entire in l and satisfies the boundary condition. It is then shown that a necessary and sufficient condition that there exist a non-trivial solution of the problem $Ly=\lambda y$, $y(0)=\theta(\lambda)(py')(0)$, for a real λ, which is in $L^2(0, \infty)$, is that F have a discontinuity at λ.

E. A. Coddington (Princeton, N.J.).

Orlov, S. A. On the theory of the resolvent of the one-dimensional regular boundary problem. Dokl. Akad. Nauk SSSR (N.S.) **111** (1956), 538–541. (Russian)

The author considers an ordinary formally self-adjoint quasi-differential expression L of order $2n$ on a finite interval $0\leq x\leq b$. Corresponding to a regular eigenvalue problem ($Ly=\lambda y$ and $2n$ linearly independent homogeneous boundary conditions) the author obtains an explicit formula for the kernel of the resolvent, which displays the constants involved in the boundary conditions. He then considers self-adjoint problems, corresponding to self-adjoint boundary conditions, and develops further properties of the kernel of the resolvent in this case. He then shows that all self-adjoint regular problems are in a one-to-one correspondence with the set of all $2n$ by $2n$ unitary matrices, and indicates how these unitary matrices occur in the resolvent. The fact that some of the boundary conditions occur at one end-point alone implies certain added information about the resolvent kernel. {Reviewer's note: Using the von Neumann Cayley transform theory, the reviewer has shown substantially these results, and more; see e.g., Ann. of Math. (2) **60** (1954), 192–211; Math. Scand. **4** (1956), 9–21, 22–28; Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 638–642; MR **16**, 133; **18**, 915.} *E. A. Coddington* (Princeton, N.J.).

Orlov, S. A. On the construction of resolvents and spectral functions of one-dimensional linear self-adjoint singular differential operators of order $2n$. Dokl. Akad. Nauk SSSR (N.S.) **111** (1956), 1175–1177. (Russian)

The author considers an ordinary formally self-adjoint differential expression L of order $2n$ on the half-axis $0\leq x<\infty$. Apparently L has real coefficients, for it is assumed that the deficiency indices of the minimal operator T_0 are equal, say equal to m. In the case considered $n\leq m\leq 2n$. The author describes all self-adjoint boundary conditions for any m, and establishes formulas for the corresponding resolvents and spectral matrices. He does this by using results on a finite interval $0\leq x\leq b$, and letting $b\to\infty$. Apparently the author is unaware of the results of the reviewer [Canad. J. Math. **6** (1954), 169–185; MR **16**, 39; and the references in the review above].

E. A. Coddington (Princeton, N.J.).

Krein, M. G. On the theory of accelerants and S-matrices of canonical differential systems. Dokl. Akad. Nauk SSSR (N.S.) **111** (1956), 1167–1170. (Russian)

In this note statements of the results of the author and others for the scalar case are extended to the matrix case. As an application the author refers to the paper of Newton and Jost [Nuovo Cimento (10) **1** (1955), 590–622; MR **17**, 155]. *N. Levinson* (Cambridge, Mass.).

Jost, Res. Eine Bemerkung über den Zusammenhang von Streuphase und Potential. Helv. Phys. Acta **29** (1956), 410–418.

The author considers the problem of what conditions on the phase will assure a given asymptotic behaviour of the potential. Use is made of the Gelfand-Levitan integral equation. *N. Levinson* (Cambridge, Mass.).

Agranovič, Z. S.; and Marčenko, V. A. Reconstruction of the potential energy from the scattering matrix. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 1(73), 143–145. (Russian)

Let y be a column vector with n components and let $y'=dy/dx$. Let $y''+\lambda^2 y=V(x)y$ on $0\leqq x<\infty$, where $V(x)$ is a matrix which satisfies $\int_0^\infty x|V(x)|dx<\infty$. Let $G(x,\lambda)$ be the matrix solution of the system satisfying $G(0,\lambda)=0$ and tending to $e^{i\lambda x}E-e^{-i\lambda x}S(\lambda)$ as $x\to\infty$ where the unitary symmetric matrix $S(\lambda)$ is the scattering matrix. Let $\mu_k=(i\lambda_k)^2$ be the bounded states of the system subject to $y(0)=0$. The matrix solution for $\lambda_k^2=-\mu_k$ is asymptotic to $e^{-\lambda_k x}M_k$ as $x\to\infty$, where M_k is a real matrix. Let

$$F(u)=\sum_k M_k M_k^* e^{-\lambda_k u}+\frac{1}{2\pi}\int_{-\infty}^{\infty}[E-S(\lambda)]e^{i\lambda u}\,d\lambda.$$

Then the kernel $K(x,y)$ which is a solution of

$$F(x+y)+K(x,y)+\int_x^\infty K(x,t)F(t+y)dt=0$$

is the kernel which determines the potential $V(x)$, since $K(x,x)=\frac{1}{2}\int_x^\infty V(t)dt$. *N. Levinson* (Cambridge, Mass.).

Moser, Jürgen. The analytic invariants of an area-preserving mapping near a hyperbolic fixed point. Comm. Pure Appl. Math. **9** (1956), 673–692.

In his work on the behavior of area-preserving mappings in the neighborhood of an unstable periodic solution of a canonical dynamical system with two degrees of freedom, G. D. Birkhoff [Acta Math. **43** (1922), 1–119] established the formal expansion of certain power series but did not determine whether these series are convergent. The present paper shows that these series actually converge.

Let the transformation (M) $x_1=f(x,y)$, $y_1=g(x,y)$, where f and g are real analytic functions defined in a neighborhood of $(0,0)$, be area-preserving and have $(0,0)$ as a hyperbolic fixed point. It can be assumed that $f(x,y)=\lambda x+\cdots$, $g(x,y)=\lambda^{-1}y+\cdots$, where λ is a real number with $\lambda^2>1$. The basic theorem proved is the following. There exists an area-preserving transformation (S) $\xi_1=\Phi(\xi,\eta)$, $\eta_1=\Psi(\xi,\eta)$, where Φ and Ψ are real analytic functions defined in a neighborhood of $(0,0)$, with $\Phi(0,0)=\Psi(0,0)=0$, such that the transform N of M by S, i.e., $S^{-1}MS=N$, is of the form (N) $u_1=U(uv)u$, $v_1=U^{-1}(uv)v$, where $U(uv)=\lambda+U_2uv+\cdots$ and this power series is convergent in a neighborhood of $(0,0)$.

As a consequence, the coefficients $U_0=\lambda$, U_2, $\cdots$ form a full system of invariants of M under the transformations S. Also, if $\lambda>0$, the mapping M can be imbedded in a family of mappings M_t depending continuously on t and such that $M_{t+s}=M_tM_s$. The actual proof of the theorem is carried through by showing its equivalence to the reduction of a system of differential equations $\dot{x}=H_y(x,y,t)$, $\dot{y}=-H_x(x,y,t)$, $H(x,y,t+1)=H(x,y,t)$ to the canonical form $\dot{\xi}=F_\eta$, $\dot{\eta}=-F_\xi$, where $F=F(\xi\eta)$ depends only on the product $\xi\eta$. *G. A. Hedlund* (New Haven, Conn.).

Tabueva, V. A. Application of the method of successive approximations to finding of separating curves. Dokl. Akad. Nauk SSSR (N.S.) **111** (1956), 301–303. (Russian)

The equation $\ddot{x}+\phi(x)\dot{x}+f(x)=0$ is considered in the xy-plane, $y=\dot{x}$. The following hypotheses are made: $f(x)$, $\phi(x)$ are of class C'; $\phi(x)\geqq m>0$; $f(0)=0$, $xf(x)>0$ for $0<x<x_1$, $xf(x)<0$ for $x_2<x<0$; $f(x_1)=0$, $f(x_2)=0$; $f'(x_1)\neq 0$, $f'(x_2)\neq 0$. It is verified that the origin is a stable equilibrium point, whereas the points $(x_1,0)$, $(x_2,0)$ are saddle-points. A sequence of functions $y_n(x)$, $0\leqq x\leqq x_1$, is defined by the equations $y_0=\{2I[f]\}^{\frac{1}{2}}$, $y_n=I[\phi+(f/y_{n-1})]$ where I denotes integration from x to x_1. It is assumed that f and ϕ satisfy the inequality $I\{\phi-I[\phi](y_1y_0)^{-1}f\}>0$ for $0\leqq x\leqq x_1$. It is then shown that the functions $y_n(x)$ converge uniformly to a solution $y=S(x)$ which is a separatrix passing through $(x_1,0)$. *W. Kaplan.*

Shapovalov, V. P. The integration of the system of two non-linear ordinary differential equations. Vestnik Leningrad. Univ. **12** (1957), no. 1, 188–196, 212. (Russian. English summary)

Let $P(x,y)$, $Q(x,y)$ be polynomials of second degree in x, y, vanishing at $(0,0)$. If P, Q are conjugate harmonic functions, the differential equations $\dot{x}=P(x,y)$, $\dot{y}=Q(x,y)$ can be written as an equation $\dot{z}=az+bz^2$ in the complex variable z; hence the solutions can be obtained by quadratures. The author describes the appearance of the solutions and determines conditions under which more general systems $\dot{x}=P(x,y)$, $\dot{y}=Q(x,y)$ can be transformed into equations of this type by linear substitutions. By further substitutions the system is related to the Riccati equation and the hypergeometric equation.

W. Kaplan (Ann Arbor, Mich.).

Otrokov, N. F. Multiple limit cycles. Mat. Sb. N.S. **41(83)** (1957), 417–430. (Russian)

In general the study of the neighborhood of a periodic solution of a differential system rests upon the construction of manifolds of section and hence upon the utilization of new auxiliary (local) coordinates. This is done here directly by means of surfaces (two dimensional case) which are loci of trajectories.

Let the system

(A) $$\dot{x}=P(x,y),\quad \dot{y}=Q(x,y)$$

have the periodic solution l: $x=\varphi(t)$, $y=\psi(t)$ (period (T)). It is assumed that P, Q are analytical in an ε_0-neighborhood $S(\varepsilon_0,l)$ of l, single-valued and with $P^2+Q^2\neq 0$ there. Let s be a segment without contact through $x_0=\varphi(0)$, $y_0=\psi(0)$ represented by $z(x=x_0+ac,\ y=y_0+bc,\ |c|\leqq c_0)$. With (A) one associates the equation

(*) $$Pf_x+Qf_y=0.$$

From a known existence theorem one infers (Theorem 1): There exist ε_0, $\delta_0>0$ such that (a) (*) possesses in $S(\varepsilon_0,l)$ a real analytical integral one of whose branches satisfies $[f_0(x,y)]_s=c$; (b) $f(x,y)=c$ determines a solution of (*) through z which, for $c<\delta_0$, intersects s once more without leaving $S(\varepsilon_0,l)$. After describing $S(\varepsilon_0,l)$ the intersection is on an arc $f^*(x,y)$ which cuts s at $\omega(c)=\sum\gamma_k c^k$ and we have $f^*=\omega(f_0)$. This transformation may have a fixed element f_0: $\omega(f_0)=f_0$. If a point of s satisfies this relation then the trajectory through it is closed. If f_0 is an n-uple root of $\omega(f_0)-f_0$ (i.e., $\gamma_1=1$, $\gamma_i=0$, $1<i<n$, $\gamma_n\neq 0$) then l is said to be an n-uple limit-cycle of (A) ($=n$-l.c.). A function $\Phi(x,y)$, analytic in $S(\varepsilon_0,l)$, is said to be single-valued of order n ($=n$-s.v.) if Φ and all its partials of order $<n$ (but not higher) are periodic and of period T on l. Theorem 2: A necessary and sufficient condition for l to be an n-l.c. is that (A) possess a first integral $f=c$ which is n-s.v.

In relation to (A) there is considered (as in questions of structural stability) a system

(B) $$\dot{x}=P+p,\quad \dot{y}=Q+q,$$

where p, q are merely analytic functions in $S(\varepsilon_0,l)$ with a

small bound for their absolute values and those of their partials up to a certain order N. Theorem 3: If l is n-l.c. for (A) then (B) will have at most n neighboring limit-cycles. Thus (A) is structurally stable around l if and only if $n=1$. For P, Q polynomials, with small polynomial derivations p, q, the author gives sufficient conditions for the appearance of two limit-cycles.

S. Lefschetz (Princeton, N.J.).

Shen, Chi-Neng. On the solution of a differential equation with nonlinearity appearing in the second derivative of combined linear and cubic terms. Quart. Appl. Math. **15** (1957), 11–30.

A non-linear feedback control is considered which is transformed to

$$\frac{d^2}{d\sigma^2}(Y+\beta Y^3)+M_n\frac{d}{d\sigma}(Y+\beta Y^3)+M_1\frac{dY}{d\sigma}+Y=r,$$

where β, M_n, M_1 and r are parameters. Setting $V=dY/d\sigma$, a first order system in the (Y, V) phase-plane is obtained. Various cases are studied. If $M_n=M_1=0$, an explicit solution is obtained. *N. Levinson* (Cambridge, Mass.).

Kato, Tosio. On linear differential equations in Banach spaces. Comm. Pure Appl. Math. **9** (1956), 479–486.

The paper is concerned with the uniqueness and the existence of the solution of a Cauchy problem (C.P.) of the equation $du(t)/dt=A(t)u(t)+f(t)$ $(a\leq t\leq b)$ in a Banach space X. Here d/dt denotes strong derivative in X, and $A(t)$ is an additive operator with a dense domain $D(A(t))$ in X to X. The results, together with a sketch of the proof, are a refinement of those of the author's previous paper [J. Math. Soc. Japan **5** (1953), 208–234; MR **15**, 437]. $A(t)$ is said to belong to the class (S') if there exists a bounded additive operator $Q(t)$ with bounded inverse such that $\tilde{A}(t)=Q(t)A(t)Q(t)^{-1}$ is the infinitesimal generator of a contraction semi-group in X. Uniqueness Theorem: The solution of the C.P. is uniquely determined by the initial value, if $A(t)$ belongs to the class (S') in such a way that $Q(t)$ is strongly continuously differentiable. Existence Theorem: Let $A(t)$ belong to the class (S') in such a way that $Q(t)$ is twice strongly continuously differentiable and the domain $D(\tilde{A}(t))$ is independent of t. Let, further, $(I-\tilde{A}(t))(I-\tilde{A}(s))^{-1}$ be strongly continuously differentiable. Then, for every strongly continuously differentiable $f(t)$, there exists a uniquely determined solution of the C.P. for every intitial value $u(a)\in D(A(a))$.

K. Yosida (Tokyo).

Ladyženskaya, O. A. On the solution of non-stationary operator equations. Mat. Sb. N.S. **39(81)** (1956), 491–524. (Russian)

On cherche $u(t)$, fonction de t (temps) à valeurs dans un espace de Hilbert H, vérifiant dans un sens plus ou moins généralisé une équation différentielle

$$(*)\qquad \frac{du}{dt}+S(t)u=f,\ u(0)\text{ et }f\text{ donnés},$$

où $S(t)$ est une famille d'opérateurs non bornés dans H, assujettis à diverses conditions de nature "elliptique". De même l'A. considère

$$(**)\qquad \frac{d^2u}{dt^2}+S(t)u=f,\ u(0),\ u'(0),\ f\text{ donnés},$$

et les opérateurs du type Schroedinger

$$(***)\qquad \frac{du}{dt}+iS(t)u=f,\ u(0)\text{ donné},$$

$S(t)$ auto-adjoint (sans hypothèse d'ellipticité pour l'existence). Excepté dans (***), $S(t)$ est de la forme $S_1(t)+S_2(t)$, où les $S_1(t)$ sont auto-adjoints à domaine indépendant de t (condition gênante pour les applications) et les $S_2(t)$ "petits" devant $S_1(t)$.

L'A. démontre des théorèmes d'existence et d'unicité par deux méthodes: (1) Méthode d'intégrale d'energie. Prenons par ex. (*), et $S(t)=S_1(t)$ pour simplifier; u est solution faible de (*) dans H, dans l'intervalle $(0, l)$, si u est de carré sommable dans $(0, l)$ à valeurs dans H et vérifie $\int_0^l[(u(t), \phi'(t)-S_1(t)\phi(t))+(f(t), \phi(t))]dt+(u(0),\phi(0))=0$ pour toute fonction ϕ continue de $[0, l]$ dans H, de dérivée de carré sommable de $[0, l]$ dans H, $\phi(t)\in$ domaine de $S_1(t)$, $t\to S_1(t)\phi(t)$ de carré sommable de $[0, l]$ dans H, $\phi(l)=0$. Le point le plus délicat est la démonstration de l'unicité; la méthode est alors celle déjà utilisée par l'A. [Le problème mixte pour une équation hyperbolique, Gostehizdat, Moscou, 1953; Dokl. Akad. Nauk SSSR (N.S.) **102** (1955), 207–210; MR **17**, 160, 161]. (2) Méthode de différence finie. d/dt est remplacé par un quotient différentiel (et dans (***) $u(t)$ par $\frac{1}{2}[u(t-h)+u(t)]$. [Pour d'autres résultats et méthodes, cf. Višik, Mat. Sb. N.S. **39(81)** (1956), 51–148; MR **18**, 215; T. Kato, J. Math. Soc. Japan **5** (1953), 208–234; MR **15**, 437; Lions, C.R. Acad. Sci. Paris **240** (1955), 390–392; MR **16**, 927; et l'article analysé ci-dessus.]

J. L. Lions.

★ **Belardinelli, Giuseppe. Operatori differenziali ipergeometrici.** Scritti matematici in onore di Filippo Sibirani, pp. 13–19. Cesare Zuffi, Bologna, 1957.

The writer studies differential operators

$$F=\sum_{n=0}^{\infty}(b_n-a_nz)z^n\frac{d^n}{dz^n},$$

where a_n and b_n are coefficients in the Newtonian interpolation series of two functions $\alpha(z)$ and $\beta(z)$; if α and β are polynomials, F is a classical hypergeometric operator. The author discusses more general conditions on α, β, and Ψ under which solutions $w(z)$ can be given for $F(w)=\Psi(z)$.

M. M. Day (Urbana, Ill.).

See also: Trigonometric Series and Integrals: Chandrasekharan. **Partial Differential Equations:** Gould. **Numerical Methods:** Zuber; Unger. **Probability:** Zinger and Linnik; Régnier. **Optics, Electromagnetic Theory, Circuits:** Arus. **Control Systems:** Higgins; Stout.

Partial Differential Equations

★ **Séminaire Schwartz de la Faculté des Sciences de Paris, 1954/1955. Equations aux dérivées partielles.** Secrétariat mathématique, 11 rue Pierre Curie, Paris, 1955. 115 pp. (polycopiées). 1200 francs.

Chapter 1 is devoted to the following theorem of Malgrange: If φ is an infinitely differentiable solution of $D\varphi=0$, $D=D(i\partial/\partial x)$ a linear partial differential operator, then φ can be approximated in any convex domain Ω by exponential polynomial solutions f. The proof proceeds by showing that a distribution T of compact support which is orthogonal to all exponential polynomial solutions f is orthogonal to all solutions φ. The condition that T be orthogonal to $\{f\}$ is shown to be equivalent to the divisibility of $\mathcal{F}(T)$, the Fourier transform of T, by the polynomial $D(\xi)$, within the ring of formal power series. Then, by a known division theorem [Séminaire H. Cartan de l'École Norm. Sup., 1951–52; MR **16**, 233] it follows that

$\mathfrak{F}(T)/D$ is entire. Next it is shown, employing an important new lemma on the quotient of entire functions and polynomials in one complex variable, that $\mathfrak{F}(T)/D$ is of exponential growth. Therefore, by the Paley-Wiener theorem (as generalized by Plancherel and Polya) $\mathfrak{F}(T)/D$ is the Fourier transform of a distribution S of compact support. From the convolution theorem of Lions [C. R. Acad. Sc. **232** (1951), 1530–1532, 1622–1624; MR **13**, 231] it follows that the support of S is also contained in Ω. Clearly, $T=D^*S$ and the orthogonality of T to all solutions follows from Green's formula.

In Chapter 2, Malgrange shows, employing the aforementioned lemma, that $y(0)$ is a continuous linear functional of D^*y in the D^{m+1} norm for all infinitely differentiable y of compact support. By the Hahn-Banach theorem this functional can be extended if necessary to all D^{m+1}; then it can be represented as $(T, D^*\phi)$. The distribution T is a fundamental solution and $g=T*f$ solves $Dg=f$ for every f of compact support. Next he constructs a solution of $Dg=f$, when the support of f is not compact, by writing f as the superposition $\sum f_i$ of such functions and putting $g=\sum g_i$, where g_i are solutions of $Dg_i=f_i$; from this a suitable entire solution of $D\phi=0$ is subtracted to make the series convergent. By a more refined division lemma, employing the L_2 norm, he shows that $Dg=f$ has a solution g which is locally L_2 if f is.

In Chapter 3 it is shown that the unrestricted solvability of $Dg=f$ is equivalent to a kind of unique continuation theorem. This is used to show that for an analytic elliptic equation with constant coefficients every solution of $D\phi=0$ defined in a domain Ω_1 can be approximated by solutions in a larger domain Ω, provided that Ω_1 does not disconnect Ω. Chapters 4–10 are devoted to the construction of fundamental solutions for polyharmonic equations and the heat equation. Chapters 11–17 contain a discussion of the real variable structure of the spaces of functions with finite m-fold Dirichlet integral, with an eye to applying these to solving boundary value problems.

P. D. Lax (New York, N.Y.).

Malgrange, Bernard. Existence et approximation des solutions des équations aux dérivées partielles et des équations de convolution. Ann. Inst. Fourier, Grenoble **6** (1955–1956), 271–355.

Since 1953, the author has been announcing interesting results in a series of notes [C. R. Acad. Sci. Paris **237** (1953), 1620–1622; **238** (1954), 196–198, 2219–2221; **240** (1955), 1958–1960; MR **15**, 626; **16**, 127; **17**, 404]. In the present monograph, he presents the detailed treatment of these, together with extensions and generalizations. It is indeed a worthy addition to the growing literature in modern differential equations in the spirit of Leray, Hörmander, Gårding, Schwartz and others. Certain of his results (especially those of Chapter I) are closely parallel to those obtained independently by Hörmander [Acta Math. **94** (1955), 161–248; MR **17**, 853] and L. Ehrenpreis [Amer. J. Math. **76** (1954), 883–903; **77** (1955), 286–292, 293–328; **78** (1956), 685–715; Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 945–946; Ann. of Math. (2) **63** (1956), 129–159; MR **16**, 834, 1123, 1122; **18**, 746; **17**, 877, 876].

The keystone of this common method is the use of the Fourier transform to convert a differential equation into a division problem for a function algebra. [If $\mathfrak{A}$ is an algebra and F and G are in $\mathfrak{A}$, then under what conditions is there $H \in \mathfrak{A}$ with $F=HG$?] The solution of this leads in turn to an inverse for the differential operator, but which may be defined only on a proper subspace of the relevant function space. If this operator can be represented there by means of a linear functional defined on the subspace, then the Hahn-Banach theorem may be applied to this functional and one may obtain an extension of the inverse operator to the whole space.

In a preliminary section, the author introduces the multitude of function spaces in which he works, and proves a number of basic lemmas. Among these, one has special importance. Let μ be a distribution with compact support ($\mu \in \mathcal{E}'$) and let D be the operator on C^∞ functions defined by $D(f)=\mu*f$. Let $V(\mu)$ be the subspace generated by the exponential polynomials

$$Q(x)=P(x_1, x_2, \cdots, x_m)\exp(\textstyle\sum_1^m \alpha_j x_j)$$

which lie in Null(D). Let $\mathfrak{F}\mu$ be the entire function which is the Fourier transform of μ. Let ν be another distribution in $\mathcal{E}'$. Then, the meromorphic function $\mathfrak{F}\nu/\mathfrak{F}\mu$ is entire if and only if $V(\mu)\subseteq V(\nu)$. This result (also obtained by Ehrenpreis) is extended to the case in which ν and μ are replaced by matrices with distribution entries; one then has $\mathfrak{F}\nu=A\cdot\mathfrak{F}\mu$, where A is a matrix with entire entries.

In Chapter I, the author deals with a general finite order partial differential operator D having constant coefficients. If $\mathbf{F}$ is a fixed function space, one seeks to characterize Null(D) and $D(\mathbf{F})$ as subspaces of $\mathbf{F}$. He first shows that, for several choices of $\mathbf{F}$, D has an "elementary solution". This is a distribution E such that $DE=\delta$. This is then applied to yield other results. Theorem 2: If $\mathbf{F}$ is $C^\infty(\Omega)$ where Ω is an open convex set, then Null(D) is the closure of the linear span of the exponential polynomials Q which it contains. He points out that the word "exponential" cannot be dropped, since Null(D) may contain no polynomials, but obtains a necessary and sufficient criterion for this in terms of an algebraic variety associated with the Fourier transform of $D\delta$. Theorem 3: With the same choice of $\mathbf{F}$, $D(\mathbf{F})=\mathbf{F}$. In particular, for any $g \in C^\infty$, the equation $D(f)=g$ has a solution in C^∞. The same results are also obtained for other function spaces. The convexity of Ω enters in because of the use of a theorem of Lions connecting the support of two distributions with that of their convolution [J. Analyse Math. **2** (1953), 369–380; MR **15**, 307]. A generalization of Theorem 3, with Ω merely open, is obtained provided that D is a local operator on $\mathcal{E}'$.

Chapter II deals with the more general operators defined by $D(f)=\mu*f$, where μ is a fixed distribution in $\mathcal{E}'$. In this, his results do not go as far as those of Ehrenpreis. As before, the null space of D is generated by the exponential polynomials which it contains. However, D is not in general an onto mapping; $D(f)=g$ does not always have a solution in the same function class as g. This, however, is true if $\mathbf{F}$ is chosen as $\mathfrak{E}$, the space of entire functions, and $\mu \in \mathfrak{E}'$. The corresponding division problem is Theorem 1: if F and G are entire functions of exponential type (m-variables) and F/G is entire, then it is also of exponential type.

In the last Chapter, the author takes up the study of general differential operators whose coefficients need not be constant. The results are stated in terms of general differentiable varieties, fibre spaces, sheaves, and currents; the results are interesting but difficult to describe briefly, without introducing the complex notation of the author. The general scope can be indicated. [See also Ehrenpreis, Proc. Amer. Math. Soc. **7** (1956), 713–718, 1131–1138; MR **18**, 584; **19**, 36.] Say that an operator

D is elliptic if Df is of class C^∞ only when f is; say that D is analiptic (=analytic-elliptic) if Df is analytic only when f is. Define the envelope of an open set $\mathcal{O}\subset\Omega$ to be the union of $\mathcal{O}$ with the relatively compact components of its complement in Ω. Say that $\mathcal{O}$ is replete if it coincides with its envelope. Say that an operator D has the Runge property on Ω if, for every replete open precompact subset $\mathcal{O}$ of Ω, the solutions of $Df=0$ on $\mathcal{O}$ which can be extended as solutions of $Df=0$ on Ω are dense in the space of all solutions of $Df=0$ on $\mathcal{O}$. The author formulates general conditions under which an operator D has the Runge property, and connects it with the general existence theorem for solutions of $D(f)=g$. Specialization: Let D be an operator with constant coefficients. D is elliptic [analiptic] if and only if D has an "elementary solution" E which is of class C^∞ [which is analytic] at the origin. If D is analiptic, then D has the Runge property, D is elliptic, and the equation $Df=g$ always has distribution solutions. The author also shows that a certain class of operators with analytic coefficients (called by him Petrowsky operators) are analiptic. There are a number of applications, using special choices of D; one obtains, for example, theorems of Behnke and Stein on holomorphic functions on analytic varieties, and theorems of de Rham dealing with harmonic functions on real varieties. The chapter concludes with a discussion of the existence of elementary kernels, in the sense of Schwartz, for general operators, proving in particular that this is the case for Petrowsky operators. *R. C. Buck.*

Lax, P. D. A stability theorem for solutions of abstract differential equations, and its application to the study of the local behavior of solutions of elliptic equations. Comm. Pure Appl. Math. **9** (1956), 747–766.

This paper consists of two parts. In the first, the author gives a new proof of the theorem that if u is a solution of the linear elliptic equation $\Delta u+Nu=0$, with N of lower order, in the neighborhood of the origin, and if $u=o(r^b)$ for every positive b as $r\to 0$, then $u\equiv 0$. A more general result with N non-linear was proved earlier by E. Heinz [Nachr. Akad. Wiss. Göttingen. IIa. **1955**, 1–12; MR **17**, 626] and Hartman and Wintner [Amer. J. Math. **77** (1955), 453–474; MR **17**, 855] and the result extended to the general second-order elliptic operator with linear highest order part by N. Aronszajn [C. R. Acad. Sci. Paris **242** (1956), 723–725; MR **17**, 854] and H. Cordes [Nachr. Akad. Wiss Göttingen, Math.-Phys. Kl. IIa. **1956**, 239–258; MR **19**, 148]. The novelty of the present proof consists of its being derived from a stability theorem for the asymptotic behaviour of solutions of differential equations in Hilbert space. {Reviewer's remarks: (1) There is an unpleasant ambiguity in the author's use of the expression "$u(t)$ tends to zero faster than any exponential". Theorem I, the stability theorem, states that if D is a linear operator in Hilbert space for which there exists an infinite sequence of lines parallel to the imaginary axis with abcissae λ_n tending to $-\infty$ on which $\|(D-\lambda)^{-1}\|\leq d^{-1}$ ($\mathrm{Re}\,\lambda=\lambda_n$) and if $\|du/dt-Du\|\leq k\|u\|$ with $k<d$, then $\|u(t)\|\geq Ae^{-bt}$ for positive constants A and b. The proof given for Theorem I shows only that it is not true that $\|u(t)\|=o(e^{-bt})$ for every positive b. The two statements are not obviously equivalent. It is not clear whether the same criticism applies to Theorem I′ of the Appendix since the complete proof is not given. It is not clear, therefore, from the arguments given, that, for the case of the solution of the elliptic equation, the uniform bound from below for the L^2-norms on small spheres is actually valid. (2) In the case that D is self-adjoint, which is the situation in the application, the condition on the resolvent is simply that there are infinitely many intervals $(\lambda_n-d, \lambda_n+d)$, $\lambda_n\to-\infty$, which are free of the spectrum of D. In this case the inequality of Lemma 1, which is the basic part of the proof, follows trivially by integration by parts without the application of the Plancherel theorem for functions with values in Hilbert space.}

In the second part of the paper, § 4, the writer shows the equivalence for solutions of $Lu=0$, with L a second-order linear elliptic operator, on bounded domains, of the unique continuation property and the Runge approximation property. The latter is said to hold if for each pair of simply connected domains $D_1\subset D_2$, every solution in D_1 can be approximated uniformly on compact subsets of D_1 by a sequence of solutions in D_2. An argument of the same type has been given for elliptic systems of arbitrary order by B. Malgrange in the paper reviewed above {Reviewer's remark: Malgrange shows only that unique continuation implies the Runge property, but his proof is essentially more general than that of the present paper, whose argument can be extended only to strongly elliptic systems.} *F. Browder* (Paris).

Louhivaara, Ilppo Simo. Über das Dirichletsche Problem für die selbstadjungierten linearen partiellen Differentialgleichungen zweiter Ordnung. Rend. Circ. Mat. Palermo (2) **5** (1956), 260–274 (1957).

The Dirichlet problem for

$$\sum_{j,k=1}^{n} D_k(a_{jk}(x)D_ju)+c(x)u=f(x)$$

on a bounded domain G is considered in the case where a_{jk} need not be definite ($D_k=\partial/\partial x_k$). The problem is taken in the sense of Courant and Hilbert [Methoden der mathematischen Physik, v. 2, Springer, Berlin, 1937, pp. 469–470]; that is, one defines S as the closure with respect to the norm

$$H(v)=\int_G\Big[\sum_{k=1}^{n}|D_kv|^2+|v|^2\Big]\,dx$$

of the set of differentiable functions vanishing near the boundary and requires, for a given function g for which $H(g)$ exists, that $u-g\in S$ and $Q(u, w)+I(f, w)=0$ for all $w\in S$. Here

$$Q(u, w)=\int_G\Big[\sum_{j,k=1}^{n}a_{jk}\,D_ju\,D_k\bar{w}-cu\bar{w}\Big]\,dx,$$

$$I(f, w)=\int_G f\bar{w}\,dx.$$

Defining S_0 as the subspace of functions $v\in S$ such that $Q(v, w)=0$ for all $w\in S$, the author points out that a necessary condition for a solution u to exist is that (*) $Q(g, v)+I(f, v)=0$ for all $v\in S_0$. He then shows that a sufficient condition for the existence of a solution is that for some $\beta>0$ (*) hold for all $v\in S_\beta^0$, the subspace of S corresponding to the spectrum contained in the interval $[-\beta, \beta]$ of the form Q with respect to the form H. The solution is constructed by orthogonal projection, using the fact that the spectrum of Q on $S\ominus S_\beta^0$ is bounded away from zero. *H. F. Weinberger* (Madison, Wis.).

McLeod, R. M.; Gergen, J. J.; and Dressel, F. G. Uniqueness of mapping pairs for elliptic equations. Duke Math. J. **24** (1957), 173–181.

Améliorant un de leurs résultats antérieurs [Gergen and Dressel, même J. **19** (1952), 435–444; Trans. Amer.

Math. Soc. **77** (1954), 151–178; MR **14**, 262; **16**, 134], les A. démontrent l'unicité de l'homéomorphisme $(x, y) \to [u(x, y), v(x, y)]$ d'un domaine donné D sur un domaine donné Δ, satisfaisant au système elliptique $\alpha u_x+\beta u_y=v_y$, $\gamma u_x+\delta u_y=-v_x$ $(0<\alpha,\ 0<\alpha\delta-\frac{1}{4}(\beta+\gamma)^2)$, et assujetti à faire correspondre 3 couples de points frontières (ou à des condition du même type), sous les hypothèses suivantes: (a) D est borné par une courbe simple de Jordan D^*, image d'un segment par $z=\varphi(t)$, la dérivée $\varphi'(t)$ étant $\neq 0$ et satisfaisant à une condition de Hölder. (b) Les coefficients α, β, γ, δ sont, ainsi que u et v, de classe C^0 sur $D^c=D\cup D^*$ et de classe C' sur D.

Ce résultat est obtenu en ramenant le problème général au cas particulier suivant: D est un cercle, $\alpha=\delta$, $\gamma=-\beta$. (L'existence de l'homéomorphisme considéré a été établie sous des conditions très générales par L. Bers et L. Nirenberg [Convegno Internazionale sulle Equazioni Lineari alle Derivate Parziali, Trieste, 1954, Edizioni Cremonese, Roma, 1955, pp. 111–140; MR **17**, 974].) *J. Lelong.*

Fava, Franco. Sulle varietà integrali del sistema:

$$\begin{cases} x_{uv}=a_1x_u+a_2x_v+a_3x_w+ax \\ x_{uu}=Lx_{vw}+b_1x_u+b_2x_v+b_3x_w+bx. \end{cases}$$

Univ. e Politec. Torino. Rend. Sem. Mat. **15** (1955–56), 121–162.

The author applies methods presented in two previous papers to a system of equations which differs only slightly from those previously considered [same Rend. **14** (1954–55), 189–237, 239–256; MR **17**, 741]. *R. T. Herbst.*

Rusak, B. I. Invariant form of generalized equations of Hamilton. Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. **16** (1955), 106–112. (Russian)

The author considers a generalization of the canonical equations of dynamics, as introduced by Raševskiĭ [Geometrical theory of partial differential equations, OGIZ, Moscow-Leningrad, 1947; MR **11**, 519]. He shows how these can be represented in explicit form in particular coordinate systems. *W. Kaplan* (Ann Arbor, Mich.).

Danilyuk, I. I. On the general elliptic system of the first order and on automorphic quasianalytical functions on surfaces. Dokl. Akad. Nauk SSSR (N.S.) **109** (1956), 253–255. (Russian)

The theory of elliptic systems of first order partial differential equations on a surface, sketched in a previous paper [same Dokl. (N.S.) **105** (1955), 11–13; MR **17**, 741], is extended to systems of the form

$$(*) \qquad a_{ik}{}^{\alpha}\frac{\partial u^i}{\partial x^{\alpha}}+a_{ik}u^i+f_k=0 \quad (i, k=1, 2, \cdots, n)$$

where $a_{ik}{}^{\alpha}$ $(\alpha=1, 2)$ is a field of n^2 contravariant vectors, Hölder continuously differentiable, the a_{ik} are Hölder continuous scalars, and an appropriate ellipticity condition is imposed. An equivalent system of integral equations is derived by making use of the generalized Cauchy integral formula developed in the paper referred to above. Also a construction is given for solutions of the system studied in that paper which are invariant with respect to a given group of transformations.

J. Cronin (New York, N.Y.).

Kreyszig, Erwin. Relations between properties of solutions of partial differential equations and the coefficients of their power series development. J. Math. Mech. **6** (1957), 361–381.

A theorem of M. Beke [Math. és Természz. Ert. **34** (1916), 1–61] can be applied directly to solutions of a linear second order elliptic partial differential equation, and results in the statement that complex-valued solutions $u(z, z_0^*)=\sum_{m=0}^{\infty}(m!)^{-1}u_m z^m$ satisfy an ordinary differential equation (of a certain type) if and only if all but a finite number of a certain set of determinants vanish. A theorem of Borel, which states that $g(z)=\sum_{m=0}^{\infty}g_m z^m$ is a rational function with denominator of degree $\leqq q$ if and only if all but a finite number of determinants of a certain collection vanish, can be applied to solutions of the elliptic partial differential equation by means of Bergman operators of the first kind, which map analytic functions into solutions of the partial differential equation. Additional theorems about sequences of coefficients are also obtained. *R. B. Davis* (Syracuse, N.Y.).

Pogorzelski, W. Etude de la solution fondamentale de l'équation elliptique et des problèmes aux limites. Ann. Polon. Math. **3** (1957), 247–284.

L'auteur considère l'équation linéaire de type elliptique

$$(*) \qquad \psi(u)\equiv\sum_{\alpha,\beta=1}^{n}a_{\alpha\beta}\frac{\partial^2 u}{\partial x_\alpha \partial x_\beta}+\sum_{\alpha=1}^{n}b_\alpha\frac{\partial u}{\partial x_\alpha}+cu=0,$$

où les $a_{\alpha\beta}$, b_α, c sont des fonctions des n coordonnées $x_1, \cdots, x_n$ dans un domaine borné mesurable Ω de l'espace euclidien à n dimensions, et résoud pour l'équation $\psi(u)=F(x_1, \cdots x_n, u)$ un problème aux limites avec la condition non linéaire sur S (surface limitant Ω),

$$\frac{\partial u}{\partial T_P}+g(P)u(P)=\Phi(P, u(P))$$

($\partial u/\partial T_P$ dérivée transversale, P point de S), énonçant le théorème d'existence sous les hypothèses suivantes: 1) $a_{\alpha\beta}$, b_α, c sont hölderiens d'exposant h dans $\Omega+S$; 2) $F(A, u)$ est hölderien dans $\Omega+S$, $|u|\leqq R$; 3) la surface fermée S vérifie des conditions de Liapounoff d'exposant h_1; 4) $g(P)$ et $\Phi(P, u)$ sont des fonctions continues vérifiant avec $F(A, u)$ (et le diamètre de Ω) certaines inégalités; et 5) unicité de la solution d'une équation intégrale vérifiée par un potentiel de simple couche correspondant à une condition limite homogène.

L'auteur démontre ensuite un théorème d'existence pour le même problème aux limites pour l'équation

$$\psi(u)=\left(Fx_1, \cdots, x_n, u, \frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_n}\right),$$

mais en supposant $g(P)$ et $\Phi(P, u)$ hölderiennes. L'auteur utilise pour ces démonstrations le théorème du point fixe de Schauder-Leray, après avoir établi des propriétés de la solution "fondamentale" de (*) (élémentaire au sens de L. Schwartz) et des potentiels de charge spatiale et de simple couche, de type hölderien, et écrit la solution u du problème posé sous forme de solution d'un système d'équations intégrales, en posant

$$u(A)=\iiint_{\Omega}\frac{\Sigma(A, B)}{\lambda_n(B)}F(B, u(B))d\tau_B+\iint_{S}\Sigma(A, Q)\varphi(Q)d\sigma_Q$$

($\Sigma(A, Q)$ solution fondamentale de (*)) et déterminant φ pour satisfaire à la condition aux limites.

Y. Fourès-Bruhat (Marseille).

Pogorzelski, W. Les propriétés d'une fonction de Green et ses applications aux équations elliptiques. Ann. Polon. Math. **3** (1956), 46–75.

Let Ω be a bounded Liapounov domain in E^n with closed boundary surface S, $F(x, u, u_1, \cdots, u_n)$ a continuous function for $x\in\Omega\cup S$, $|u|\leqq R$, $|u_j|\leqq R$, satisfying a

Hölder condition jointly in all its variables. Let $M_F = \sup |F|$. Suppose that there exists no non-null solution of $\Delta u=0$ in Ω satisfying the boundary condition $\partial u/\partial n + a(s)u=0$ on S, where $a(s)$ is a given function satisfying a Hölder condition on S. Then the author shows by an application of the Schauder fixed point theorem that for M_F sufficiently small there exists a solution of $\Delta u = F(x, u, du/\partial x_1, \cdots, \partial u/\partial x_n)$ in Ω satisfying $\partial u/\partial n + a(s)u=0$ on S. {Reviewer's remark: the author's theorem as stated on page 75 omits the condition that M_F is small, but the latter is essential to the proof.} *F. Browder* (Paris).

Bers, Lipman. Remark on an application of pseudo-analytic functions. Amer. J. Math. **78** (1956), 486–496.

The local behaviour of solutions of the equation

$$L\varphi = a_{11}\varphi_x + 2a_{12}\varphi_{xy} + a_{22}\varphi_{yy} + a_1\varphi_x + a_2\varphi_y + a_0\varphi = 0$$

is studied near a regular or an isolated singular point. All coefficients and solutions are defined on a fixed neighborhood of the origin in the z-plane, the equation is elliptic, and it is assumed that $a_{11}=a_{22}=1$, $a_{12}=0$ at $z=0$. Two kinds of smoothness hypotheses are made: (α) a_{ij} satify a Hölder condition, a_i lie in L^p for some $p>2$; (β) all coefficients satisfy a Hölder condition. A solution φ is said to have a zero of nth order at zero if for some complex $k\neq 0$, $\varphi \sim \mathrm{Re}(kz^n)$, $\varphi_x - i\varphi_y \sim nkz^{n-1}$. In case of hypothesis (β), it is also required that $\varphi_{xx} - i\varphi_{xy} \sim -\varphi_{yy} - i\varphi_{xy} \sim n(n-1)kz^{n-2}$. The author shows that under hypotheses (α) or (β) every solution φ which vanishes at the origin, and is not identically zero, has a zero of some integer order at the origin. If φ is a solution on a deleted neighborhood of zero, φ is said to have a logarithmic singularity if for some $k\neq 0$, $\varphi \sim k \log |z|$, $\varphi_x - i\varphi_y \sim kz^{-1}$. φ is said to have a pole of order n at $z=0$ if $\varphi \sim \mathrm{Re}(kz^{-n})$, $\varphi_x - i\varphi_y \sim -nkz^{-n-1}$. The origin is said to be an essential singularity of φ if, for every $N>0$, $\limsup(|z|^N\varphi) = \limsup(-|z|^N\varphi) = \limsup(|z|^N|\varphi_x - i\varphi_y|) = +\infty$. The author proves that every singularity of φ which is isolated at zero is either removable, logarithmic, polar, or essential. The proofs use an inequality of Calderón and Zygmund [Acta Math. **88** (1952), 85–139; MR **14**, 637] to reduce the given problem to one in the author's theory of pseudo-analytic functions. *F. Browder* (Paris).

Ericksen, J. L. On the Dirichlet problem for linear differential equations. Proc. Amer. Math. Soc. **8** (1957), 521–522.

The system of linearized equations of non-linear elasticity theory is of the form

$$\sum_{k=1}^{n} (\lambda+\mu)\frac{\partial^2 u_k}{\partial x_k \partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_k^2} = 0 \quad (i=1, \cdots, n)$$

with parameters λ and μ and dependent variables $(u_1, \cdots, u_n)$. The system is elliptic if $(\lambda+2\mu)\mu \neq 0$, strongly elliptic if $(\lambda+2\mu)\mu > 0$. In the elliptic, non-strongly elliptic case, Bicadze [Uspehi Mat. Nauk (N.S.) **3** (1948), no. 6(28), 211–212; MR **10**, 300] gave an example in which the uniqueness of the Dirichlet problem fails, by taking $n=2$, $\lambda=-3\mu$, u_2 replaced by $2^{\frac{1}{2}}u_2$. In the present paper it is shown more generally, by an elementary argument, that if $(\lambda+2\mu)\mu<0$, the system will admit non-zero solutions, all of whose components vanish on the boundary of some region R of any prescribed diameter and are analytic in the closure of R. *F. Browder* (Paris).

Stummel, Friedrich. Singuläre elliptische Differentialoperatoren in Hilbertschen Räumen. Math. Ann. **132** (1956), 150–176.

Let A be the operator in $L^2(E^n)$ whose domain is the family of twice-continuously differentiable functions having compact support in E^n and which is defined by

$$Au = -\Delta u + 2i\sum_k a_k \partial u/\partial x_k + i\sum_k (\partial a_k/\partial x_k)u + bu,$$

where the a_k ($k=1, \cdots, n$) and b are given functions on E^n. If the $a_k \in C^1(E^n) \cap L^1(E^n)$, b is locally L^2, and there exists $\alpha>0$ so that the integral $\int_{|x-y|\leq 1} |b(y)|^2 |x-y|^{4-\alpha-n} dy$ is uniformly bounded on every compact subset of E^n, then the adjoint A^* of A is characterized by a mean-value formula. Results are established on the essential self-adjointness of the symmetric operator A under various hypotheses on the coefficients. In particular, if all the a_k vanish identically while b satisfies the conditions above, it is shown that A is essentially self-adjoint, extending previous results of Carleman [Ark. Mat. Astr. Fys. **24B** (1934), no. 11], Friedrichs [Math. Ann. **109** (1934), 465–487, 685–713] and Kato [Trans. Amer. Math. Soc. **70** (1951), 195–211; MR **12**, 781]. It is shown that the essential self-adjointness of A is not altered by the addition of terms of the form $q(x)u$, where q satisfies certain boundedness conditions. The Coulomb potential for several particles and the Stark-effect potential are also treated. *F. Browder* (Paris).

Walter, Wolfgang. Mittelwertsätze und ihre Verwendung zur Lösung von Randwertaufgaben. Jber. Deutsch. Math. Verein. **59** (1957), Abt. 1, 93–131.

The author solves the boundary value problem for the equation $\Delta^p u=0$ in an n-dimensional region G by reducing it to the variational problem $D(u)=\min.$, where D is the corresponding Dirichlet integral. A minimizing sequence u_n is smoothed out by the application of the means

$$M(u, x, R, g) = \frac{1}{V(R)} \int_{|y-x|\leq R} g\left(\frac{|y-x|}{R}\right) u(y)dy,$$

where $V(R)$ is the volume of the ball $|y-x|\leq R$ and $g(t)$ a function in $0\leq t\leq 1$. The main point is that if $g(t)$ is bounded and satisfies $I_i = n\int_0^1 g(t)t^{n+2i-1}dt = 1$ for $i=0$, $I_i=0$ for $i=1, 2, \cdots, p-1$, and $I_p \neq 0$, then the relation $M(u, x, R, g)=u(x)$ for a locally integrable function $u(x)$, $x \in G$, is equivalent to $\Delta^p u=0$. Under some further restrictions on g it is possible to show that the means $M(u_n, x, R, g)$ converge (uniformly on compact subsets of G) to the solution of the problem. The author also treats equations of type $\Delta^p u + q(x)u = F(x)$. *G. G. Lorentz* (Detroit, Mich.).

Kudryavcev, L. D. On the solution by the variational method of elliptic equations which degenerate on the boundary of the region. Dokl. Akad. Nauk SSSR (N.S.) **108** (1956), 16–19. (Russian)

Campanato, Sergio. Sui problemi al contorno relativi al sistema di equazioni differenziali dell'elastostatica piana. Rend. Sem. Mat. Univ. Padova **25** (1956), 307–342.

Il secondo problema dell'elastostatica piana consiste nel ricercare, in un dominio piano D, una soluzione $u=(u_1, u_2)$ del sistema di equazioni dell'elasticità piana verificante la condizione $L(u)=t$, dove t è un vettore assegnato su FD e $L(u)$ è il vettore degli sforzi. L'A. dimostra che, se t verifica certe condizioni di compatbilità, il problema ammette una soluzione appartenente alla classe $\{u\}$ dei potenziali elastici del tipo (*) $u(Q) = \int_{FD} d(Q) * S(P, Q) ds_Q$ a densità $d(Q)$ sommabile, $S(P, Q)$ essendo la matrice di Somigliana. Nei confronti di precedenti risultati di G. Fichera [Ann. Scuola Norm. Sup. Pisa (3) **4** (1950), 35–99;

MR 13, 42] e N. I. Muskhelishvili [Alcuni problemi fondamentali della teoria matematica dell'elasticità, 3° ed., Izdat. Akad. Nauk SSSR, Moscow-Leningrad, 1949; MR 11, 626; 15, 370] la trattazione dell'A. ha il pregio di una maggiore economia di ipotesi sul vettore t, che viene supposto soltanto sommabile su FD. Di più si perviene al risultato traducendo il problema in un sistema di equazioni integrali non singolari. Fondamentale a questo riguardo è un risultato preliminare dell'A., secondo il quale ogni vettore della classe $\{u\}$ oltre che della rappresentazione (*) è suscettibile anche di un'altra rappresentazione dello stesso tipo in cui alla matrice $S(P, Q)$ ne subentra un'altra $F(P, Q)$ costruita in modo che il vettore $L(u)$ risulti esprimibile mediante integrali ordinari. Questo risultato, che può considerarsi come la generalizzazione di un teorema di E. Magenes [stessi Rend. 24 (1955), 510–522; MR 19, 262] relativo al caso di un'equazione ellittica in una sola funzione incognita, permette anche di dimostrare che la classe $\{u\}$ non si differenzia da quella considerata da G. Fichera [Rend. Sem. Fac. Sci. Univ. Cagliari 18 (1948), 1–22; MR 11, 700] delle soluzioni delle equazioni dell'elasticità piana per cui sussistono la prima e la seconda formula di Green. L'A. considera infine anche il problema misto che consiste nell'assegnare su una porzione di FD il vettore u e sulla rimanente parte $L(u)$ e dimostra per tale problema dapprima dei teoremi di unicità e in un secondo momento dei teoremi di completezza di quei sistemi di funzioni che, nell'applicazione di un noto metodo di M. Picone [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 2 (1947), 365–371, 485–492, 717–725; MR 9, 145, 286, 287], intervengono nella traduzione del problema in un sistema di equazioni di Fischer-Riesz. In proposito vedi anche il lavoro di G. Fichera citato per ultimo. *C. Miranda* (Napoli).

Lasota, A. Sur une généralisation d'un problème de Z. Szmydt concernant l'équation $u_{xy}=f(x, y, u, u_x, u_y)$. Bull. Acad. Polon. Sci. Cl. III. 5 (1957), 15–18, III. (Russian summary).

The Szmydt problem consists in finding a vector function $U(x, y)$ which satisfies in the rectangle R $(0\leqq x\leqq a, 0\leqq y\leqq b)$, the differential equation

$$U_{xy}=f(x, y, U, U_x, U_y)$$

with the assigned boundary conditions: (a) $U_x=g(x, U_y)$ on a given curve $y=k(x)$, (b) $U_y=h(y, U_x)$ on $x=l(y)$, and (c) at a point (x_0, y_0) in R, $U(x_0, y_0)=A_0$, a constant vector. In this note the author uses the same method (Schauder's fixpoint theorem) to solve the more general problem when, in conditions (a) and (b), the functions g, h, k and l may depend on U. These four functions and f are assumed to be continuous in R and for arbitrary values of the other arguments; f satisfies a Lipschitz condition with respect to U_x, U_y; k and l with respect to U; g with respect to U_y; h with respect to U_x. In order to prove the existence theorem, four simple inequalities are required to hold among the Lipschitz constants, the size of R and the maxima of $|f|$, $|g|$ and $|h|$. *Yu Why Chen.*

Wintner, Aurel. On the conditions of validity of Riemann's method of integration. Quart. Appl. Math. 15 (1957), 94–98.

In the classical Riemann's formula for solutions of an equation

$$(*)\qquad u_{xy}+a(x, y)u_x+b(x, y)u_y+c(x, y)u=0,$$

the Riemann function is obtained by solving the adjoint equation which has coefficients involving the derivatives of a and b. In this note the adjoint equation is represented in integrated form, and the existence of a unique continuous solution is proved for continuous boundary data. The Riemann function obtained in this manner can be used in the classical formula to solve (*) for continuous differentiable values on the boundary. *Yu Why Chen.*

Halilov, Z. I. On the method of expansion in eigenfunctions of the principal part of the equation in the solution of mixed problems. Dokl. Akad. Nauk Azerbaĭdžan. SSR. 10 (1954), 235–239. (Russian. Azerbaijani summary)

Halilov, Z. I. On the method of expansion in eigenfunctions of the principal part of the equation in the solution of mixed problems. Azerbaĭdžan. Gos. Univ. Trudy Ser. Fiz.-Mat. 4 (1954), 5–33. (Russian. Azerbaijani summary)

The author considers the problem of finding the solution of the equation

$$Lu\equiv\frac{\partial^2 u}{\partial x^2}+C(t,x)\,\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial t^2}=0$$

with the conditions $u|_{x=0}=u|_{x=\pi}=0$ and $u|_{t=0}=\phi(x)$, $\partial u/\partial t|_{t=0}=\psi(x)$. The function u is sought in the form

$$(1)\qquad u(t, x)=\sum_{m=1}^{\infty} A_m(t)\sin mx.$$

In order to determine the coefficients $A_m(t)$ we have the system

$$(2)\qquad A_m(t)=\sum_{n=1}^{\infty}\int_0^t K_{mn}(t, \tau)A_n(\tau)d\tau+f_m(t),$$

where K_{mn} and f_m are known functions. Under certain conditions on the data of the problem, the system (2) has a unique solution, and the series $\sum_{m=1}^{\infty}|A_m|$, $\sum_{m=1}^{\infty}|dA_m/dt|$ are uniformly convergent for $t\in[0, T]$. Similarly it is found that $u_N=\sum_{m=1}^{N}A_m(t)\sin mx$ and $\partial u_N/\partial t$ converge uniformly in x and t to certain functions u and $\partial u/\partial t$, respectively. Moreover, $L[u_N]\to 0$.

The function u, obtained by this passage to the limit, is called by the author, after S. L. Sobolev, a generalized solution. Theorem II, proved by the author for equation (1) (theorem of uniqueness of the classical solution), is known for general hyperbolic equations without any limitations on the signs of the coefficients in the lower derivatives (the author demands that $C(t, x)\leqq 0$). In Theorem III it is proved that if the classical solution exists then it coincides with (1), and not in general with the generalized solution, as the author asserts. The uniqueness of the generalized (in the above sense) solution is not proved by the author.

The author is evidently not acquainted with the literature dealing with the mixed problem for hyperbolic equations with coefficients depending on $x_1, \cdots, x_n, t$. In it stronger theorems are established for existence and uniqueness for general linear hyperbolic equations.

O. A. Ladyženskaya (RŽ Mat. 1955, no. 233).

Šil'krut, D. I. On a problem of heat conduction for two media. Prikl. Mat. Meh. 20 (1956), 284–288. (Russian)

Let $\theta_1(x, t)$ and $\theta_2(x, t)$ represent the temperatures in two media which are separated by the plane $x=0$. These

functions must satisfy the heat equations

$$\frac{\partial\theta_1}{\partial t}=\frac{\partial}{\partial x}\left[k_1(x)\frac{\partial\theta_1}{\partial x}\right] \quad (x\geqq 0,\ t\geqq 0),$$

$$\frac{\partial\theta_2}{\partial t}=\frac{\partial}{\partial x}\left[k_2(x)\frac{\partial\theta_2}{\partial x}\right] \quad (x\leqq 0,\ t\geqq 0),$$

where k_1 and k_2 are the coefficients of heat conduction in the two media. The initial temperatures are $\theta_1(x, 0)=\theta_2(x, 0)=0$. The continuity of the temperature requires that $[\theta_1(x, t)]_{x=0+}=[\theta_2(x, t)]_{x=0-}$ and the condition of thermal equilibrium is taken to be

$$(*)\quad -\lim_{x\to 0+}\left[\lambda_1(x)\frac{\partial\theta_1}{\partial x}\right]+\lim_{x\to 0-}\left[\lambda_2(x)\frac{\partial\theta_2}{\partial x}\right]+h\theta_1=w(t).$$

Finally, it is required that $\lim_{x\to\infty}\theta_1=0$, $\lim_{x\to-\infty}\theta_2=0$. This problem has been solved previously by E. M. Dobryšman [Prikl. Mat. Meh. **18** (1954), 219–224; MR **16**, 45], but by a different technique which is applicable for small t. In this paper, the methods of the operational calculus are used to obtain a solution in terms of contour integrals which is valid for any finite t. It is indicated that the function $w(t)$ in (*) may be replaced by the impulse function $\delta(t)$ and the method is still applicable.

C. G. Maple (Ames, Ia.).

Kolodner, I. I. Free boundary problem for the heat equation with applications to problems of change of phase. I. General method of solution. Comm. Pure Appl. Math. **9** (1956), 1–31.

This paper considers the free boundary problem for the heat equation which is encountered in the change of phase process in several physical problems. Such problems arise in the (a) motion of a plane liquid vapor interface (which may or may not be in a confined medium); (b) decay by evaporation (or growth by condensation) of a liquid drop; (c) dissolution of a gas bubble in liquid; (d) freezing of water; and (e) solidification of the terrestrial crust. Each of these problems is discussed in a section on applications.

The problem is formulated as follows: let an open simply-connected domain D_R in the x, t-plane be bounded by a segment BC of the characteristic $t=\bar{t}$, a given arc AC and an unknown arc (free boundary) AB whose equation is $x=R(t)$. The problem is to find $R(t)$, $0\leqq t\leqq\bar{t}$, and $U(x, t)$ satisfying

$$LU\equiv U_{xx}-U_t=0 \text{ for } (x, t)\in D_R,\ \Gamma U=0 \text{ on arc } AC,$$

$$U(R, t)=f(t) \text{ for } 0\leqq t\leqq\bar{t},\ U_x(R, t)=g(t) \text{ for } 0\leqq t\leqq\bar{t}.$$

The functions f and g are not required to be explicit functions of t but may be functionals of R, with f, g, and R satisfying certain class properties. The nature of the boundary operator Γ is required to be such that the following problem for the determination of $w(x, t)$ has a unique solution:

$$Lw=0 \text{ for } (x, t)\in D,\ \Gamma(w+v)=0 \text{ on } AC,$$

$$w(x, 0)=0\ (x<a),\ |w(x, t)|\leqq A,\ |t^b w_x(x, t)|\leqq A.$$

The domain D is the set of points lying to the left of AC, v is a differentiable function along AC, and a and A are constants arising from the restrictions on f, g and R.

The essential feature of the method used is based on the fact that for this type of problem it is possible to eliminate the solution of the differential equation, and in the process obtain a functional equation which the unknown boundary function $R(t)$ must satisfy. This functional equation is a non-linear integro-differential equation, which in some of the physical problems mentioned above is of the Volterra type of the second kind. In the process of obtaining the functional equation, a representation of $u(x, t)$ in terms of the boundary function is obtained, so that once the boundary function is known, the solution to the problem may be found.

C. G. Maple (Ames, Ia.).

McKean, Henry P., Jr. Elementary solutions for certain parabolic partial differential equations. Trans. Amer. Math. Soc. **82** (1956), 519–548.

Let S be an open interval (s_1, s_2), not necessarily bounded; let $c(x)$ be continuous and $\leqq 0$ on S; let $m(dx)$ be a Borel measure, strictly positive on open subsets of S; and let B be the second order differential operator $Bu=u^+(dx)/m(dx)+c(x)u(x)$ as introduced by W. Feller. The operator B, considered as an operator from $C(S)$ into $C(S)$ contracted by a classical side condition (such as $\frac{1}{2}u(s_1)+\frac{3}{2}u^+(s_1)=0$ and $u(s_2)=0$, or the like), is denoted again by B. Let $L_2(S)$ be the space of m-measurable functions v on S such that $\|v\|_2^2=\int_S|v(s)|^2m(ds)<\infty$. The eigendifferential expansion of the operator B in the Hilbert space $L_2(S)$ is constructed, similarly as in the case of the ordinary Sturm-Liouville operator and as completed by Weyl, Stone, Titchmarsh and Kodaira. The construction is performed by approximating the appropriate Green operator for $(\mu-B)u=v$ by suitable compact Green operators. The result is applied to compute the inversion

$$p(t, x, s)=(2\pi i)^{-1}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty}\exp(t\mu)G(x, s, \mu)d\mu \quad (t, \varepsilon>0)$$

and to obtain the elementary solution $p(t, x, s)$ of $u_t(t, s)=Bu(t, s)$ $(t>0)$ with the properties: i) $p(t, x, s)=p(t, s, x)$ is positive and $\int_S p(t, x, s)m(ds)\leqq 1$; ii) $\partial^n p(t, \cdot, s)/\partial t^n$ satisfies the side conditions and equals $B^n p(t, \cdot, s)$; iii) the operator S_t defined by

$$(S_t u)(x)=\int_S p(t, x, s)u(s)m(ds)$$

constitutes a semi-group on $C(S)$ into $C(S)$ satisfying $\partial^n(S_t u)(x)/\partial t^n=B^n(S_t u)(x)$ for $t>0$, $u\in C(S)$ and $n>0$.

K. Yosida (Tokyo).

Mizohata, Sigeru. Le problème de Cauchy pour les équations paraboliques. J. Math. Soc. Japan **8** (1956), 269–299.

On considère le système différentiel

$$(*)\quad \frac{\partial}{\partial t}U(t)=A(x, t, \partial/\partial x)U(t)+F(t)=A(t)U(t)+F(t),$$

où $U=(u_0, u_1, \cdots, u_{m-1})$, $A(x, t, \partial/\partial x)$ matrice m, m; on suppose que ce système est p-parabolique au sens de Petrowsky [Bull. Univ. d'Etat Moscou. Sér. Internat. Sect. A. **1** (1938), no. 7]. Les coefficients de A sont supposés indéfiniment différentiables, bornés ainsi que toutes leurs dérivées. On veut résoudre le problème de Cauchy, posé dans les espaces suivants: on désigne par D_q l'espace des vecteurs $(f_1, \cdots, f_m)$ tels que f_i admette une transformée de Fourier $\hat{f}_i$ vérifiant: $(1+|\xi|)^{q-(i-1)p}\hat{f}_i$ est de carré sommable, q entier ou non. On donne alors dans (*) la fonction $F(t)$ continue de $t\geqq 0$ dans D_{q+p}, et on cherche $U(t)$ continue de $t\geqq 0$ dans D_{q+p}, une fois continûment différentiable de $t\geqq 0$ dans D_q, $U(0)$ étant donné dans D_{q+p}. Le résultat est que ce problème admet une solution unique (q quelconque). Ce résultat n'est pas essentiellement nouveau [cf. Ėĭdelman, Mat. Sb. N.S.

38(80) (1956), 51–92; MR 17, 857] mais la méthode de démonstration est nouvelle et élégante. Il s'agit d'une adaptation au cas parabolique de la méthode de Leray relative aux opérateurs hyperboliques. (1) Le point essentiel est la construction d'une matrice $B(t)$, opérateur définissant une norme équivalente à la norme naturelle sur D_q et telle que

$$B(t)A(t)+(B(t)A(t))^* \leqq -\alpha E_{q+p'} \quad (\alpha>0,\ p'=p/2),$$

où E_p est défini de façon générale par transformation de Fourier inverse de la matrice diagonale

$$\| |\xi|^{2l-2(l-1)p} \|.$$

La construction de $B(t)$ occupe la partie I de l'article et utilise naturellement la p-parabolicité de (*). (2) On utilise ensuite (1), le fait que les $B(t)$ dépendent convenablement de t, et une variante du théorème de Kato [J. Math. Soc. Japan 5 (1953), 208–234; MR 15, 437] pour conclure.

Généralisation en remplaçant les D_s par des espaces de distributions tempérées [cf. Schwartz, Théorie des distributions, t. 1, 2, Hermann, Paris, 1950, 1951; MR 12, 31, 833]. *J. L. Lions* (Nancy).

Nakamura, Mikio. On the linear partial differential equation of second order in n independent variables with constant coefficient. J. Gakugei Tokushima Univ. Nat. Sci. Math. 6 (1955), 51–62.

In this paper a general integral is found for equations of the form $P(D)w=f(x_1, \cdots, x_n)$, where $P(D)$ is a linear differential operator of second order with constant coefficients, which can be factored into the product of two first order operators, either distinct or identical. When $P(D)$ cannot be factored, it is reduced, by suitable transformations, to a canonical form for which a complete integral is found. Proofs are given in detail and there are illustrative examples. *D. L. Bernstein.*

Žautykov, O. A. On the problem of construction of a solution of a nonlinear partial differential equation of the first order with a countable number of independent variables. Izv. Akad. Nauk Kazah. SSR. Ser. Mat. Meh. **1956**, no. 5(9), 45–61. (Russian)

Let

$$(*) \qquad p=f(t, x_1, x_2, \cdots, z, p_1, p_2, \cdots),$$

where $p=\partial z/\partial t$, $p_i=\partial z/\partial x_i$, and assume that in the region H defined by $|t-t_0|\leqq a$, $|x_s|\leqq R$, $|z|\leqq C$, $|p_s|\leqq N$, the function f is continuous and bounded, has continuous bounded first and second partials, and if $(x', p')=(x_1', \cdots, p_1', \cdots)$ and (x'', p'') are two points such that $x_1'=x_1''$, $\cdots$, $x_{m-1}'=x_{m-1}''$, $p_1'=p_1''$, $\cdots$, $p_{m-1}'=p_{m-1}''$, and $\Delta u=\sup\{|x_s'-x_s''|, |p_s'-p_s''|\}$, then

$$|f(t, x', z, p')-f(t, x'', z, p'')| \leqq \varepsilon_m \Delta u,$$

where $\varepsilon_m \to 0$ as $m\to\infty$ uniformly in t, and similarly for f_{x_s}, f_{p_s} and f_z; moreover, if (x', z', p'), (x'', z'', p'') are any two points and $\Delta v=\sup\{|x_s'-x_s''|, |p_s'-p_s''|, |z'-z''|\}$, then $\sum |\Delta f_{x_s}|+\sum |\Delta f_{p_s}|+|\Delta f_z| \leqq A\Delta v$, the Δ's representing increments of the corresponding functions from one point to the other. These assumptions warrant the differentiability of composite functions (provided $x_k(t)$ and $p_k(t)$ have bounded derivatives) and the uniform convergence of the corresponding series. Then (*) has a unique continuously differentiable solution in a region SCH which coincides with a given function $z=\omega(x_1, x_2, \cdots)$ for $t=t_0$. *J. L. Massera* (Montevideo).

Lavrent'ev, M. A.; and Šabat, B. V. Geometrical properties of solutions of non-linear systems of partial differential equations. Dokl. Akad. Nauk SSSR (N.S.) **112** (1957), 810–811. (Russian)

The authors consider pairs of non-linear partial differential equations of the first order

$$F_i(x, y, u, v, u_x, u_y, v_x, v_y)=0 \quad (i=1, 2).$$

Lavrent'ev [Mat. Sb. **21(63)** (1947), 285–320; Izv. Akad. Nauk SSSR. Ser. Mat. **12** (1948), 513–554; MR **10**, 290; **11**, 650] had previously defined the concept of a strongly elliptic non-linear system. The authors give a slight generalization of this definition, and call a solution of such a system a quasi-conformal map associated with this system. The following theorems are stated. (I) For any Riemann surface of hyperbolic type, and any strongly elliptic system, there exists a homeomorphic quasi-conformal map f associated with this system of the surface onto the unit circle. (II) Any differentiable solution $f=u+iv$ of a strongly elliptic system in a domain D for which $u\neq$const, $v\neq$const, gives a homeomorphic map of D onto a certain Riemann surface. (III) Any differentiable solution $f=u+iv$ of a strongly elliptic system in a domain D for which $u\neq$const, $v\neq$const can be represented as the superposition $f=F\circ T$ (i.e. $f(z)=F(T(z))$, where F is an analytic function, and T is a homeomorphic quasi-conformal map of D onto itself. *E. A. Coddington.*

Halilov, Z. I. Solution of Cauchy's problem for an infinite system. Akad. Nauk Azerbaidžan. SSR. Trudy Inst. Fiz. Mat. **4–5** (1952), 5–19. (Russian. Azerbaijani summary)

Košelev, A. I. Existence of a generalized solution of the elastic-plastic problem of torsion. Dokl. Akad. Nauk SSSR (N.S.) **99** (1954), 357–360. (Russian)

In a plane region Ω of the cross-section of a bar, which is bounded and simply connected and whose boundary Γ is three times continuously differentiable, a generalized solution (for the definition of this term see same Dokl. **91** (1953), 1263–1266; MR **15**, 711) is sought for the equation of elastic-plastic torsion

$$(*) \qquad P(u)=\frac{\partial}{\partial x}\left[f(T^2)\frac{\partial u}{\partial x}\right]+\frac{\partial}{\partial y}\left[f(T^2)\frac{\partial u}{\partial y}\right]+\omega=0$$

where $T^2=(\partial u/\partial x)^2+(\partial u/\partial y)^2$ and ω is a constant, and the solution must satisfy the boundary condition $u|_\Gamma=0$.

It is assumed that the given function f has the form $f(t)=f_0+\phi(t)$, where f_0 is a positive constant and $\phi(t)$ does not vanish, is three times continuously differentiable and satisfies the inequality $\phi(t)\geqq bt^r$ for $t\geqq 0$; $b>0$, $r>0$.

This problem is written in the form of a functional equation $p(u)=0$, generated by the elliptic equation (*). The author begins by considering the supplementary equation

$$P_\lambda(u)=\frac{\partial}{\partial x}\left[(f_0+\lambda\phi)\frac{\partial u}{\partial y}\right]+\frac{\partial}{\partial y}\left[(f_0+\lambda\phi)\frac{\partial u}{\partial y}\right]+\omega=0,$$

where λ is a numerical parameter. For $\lambda=0$ this is equivalent to a linear Dirichlet problem and for $\lambda=1$ to the original problem. By making use of the conditions of convergence for Newton's method obtained by L. V. Kantorovič [same Dokl. **59** (1948), 1237–1240; MR **9**, 537] it is shown that for sufficiently small λ the supplementary problem has a generalized solution.

Further, an operator $P(u, \lambda)$ is considered, which carries a space X of type B into a space Y of type B

and depends on the numerical parameter $\lambda(0\leq\lambda\leq 1)$. The lemma is proved that if, under certain well-defined conditions, the equation $P(u, \lambda)=0$ has a solution for $\lambda=0$, then it also has a solution for $\lambda=1$. Since these conditions are fulfilled for the supplementary problem, the proof of existence of a solution for the original problem is complete.

There is a bibliography with six entries.

G. S. Šapiro (RŽ Meh **1955**, no. 5653).

★ **Gould, S. H. Variational methods for eigenvalue problems. An introduction to the methods of Rayleigh, Ritz, Weinstein, and Aronszajn.** Mathematical Expositions No. 10. University of Toronto Press, Toronto, 1957. xiv+179 pp. $6.00.

Mathematical literature has long been concerned with the characterization of eigenvalues as minima of integrals, a topic of importance in many branches of pure and applied mathematics. The purpose of this book is to give an exposition of variational methods for the computation of eigenvalues. The standard variational method is the Rayleigh-Ritz method, which gives upper bounds for the eigenvalues of differential problems. In the late thirties, Alexander Weinstein developed a corresponding method for the determination of lower bounds [see, e.g., Étude des spectres des équations aux dérivées partielles de la théorie des plaques élastiques, Gauthier-Villars, Paris, 1937]. Later, N. Aronszajn [Proc. Nat. Acad. Sci. U.S.A. **34** (1948), 474–480, 594–601; MR **10**, 382] found that Weinstein's method is a counterpart of a method which is more general than the one of Rayleigh-Ritz, and which was developed by Aronszajn himself under the name of the generalized Rayleigh-Ritz method. Successful numerical applications of these methods have been made over a period of twenty years. Of course, as is well known, several other methods have been devised to supplement the classical Rayleigh-Ritz procedure. However, the methods discussed in this book possess an inherent symmetry, which is perhaps best revealed by their presentation in Hilbert space. While the numerical successes of these methods attracted the attention of engineers, the abstract mathematical structure appealed to pure mathematicians [see, e.g., Weyl, Bull. Amer. Math. Soc. **56** (1950), 115–139; MR **11**, 666]. The present book, dedicated to Weinstein, constitutes the first account in book form in English of his method. One finds here, also for the first time in book form, a detailed account of the work of N. Aronszajn and his colleagues, which had hitherto only appeared, for the most part, in the form of technical reports.

The underlying ideas behind these various fundamental methods for the approximation of eigenvalues will now be outlined. For definiteness, consider the problem of the clamped vibrating plate:

$$(*)\qquad \Delta\Delta w-\lambda^2 w=0,\ \text{on } D,\qquad w=\frac{\partial w}{\partial n}=0,\ \text{on } C,$$

where D is a bounded plane open connected set with a smooth boundary C. Let $0<\lambda_1^2\leq\lambda_2^2\leq\lambda_3^2\leq\cdots$ be the eigenvalues of (*). In terms of the scalar product $(f, g)=\iint_D f(x, y)g(x, y)dxdy$, their variational definition is

$$(**)\qquad \lambda_n^2(A)=\max_{\phi_1,\cdots,\phi_{n-1}}\ \min_{\substack{w\in A\\(w,\phi_i)=0\\ i=1,\cdots,n-1}}\frac{(\Delta w, \Delta w)}{(w, w)}\quad (n=1, 2, 3, \cdots),$$

where the arbitrary real-valued functions ϕ_i are defined on $D+C$; A is the set of admissible functions, i.e., those satisfying the boundary conditions $w=\partial w/\partial n=0$ on C; and the eigenvalues λ_n^2 of (*) have been designated by $\lambda_n^2(A)$, in order to emphasize their explicit dependence on the set A. Let S and B be sets of functions such that $S\subset A\subset B$, and let $\lambda_n^2(S)$ and $\lambda_n^2(B)$ denote the corresponding eigenvalues obtained by replacing A by S and by B in (**). Then it follows that $\lambda_n^2(B)\leq\lambda_n^2(A)\leq\lambda_n^2(S)$ $(n=1, 2, \cdots)$.

It is now clear how to obtain a monotone nondecreasing sequence of lower bounds and a monotone nonincreasing sequence of upper bounds for $\lambda_n^2=\lambda_n^2(A)$. For if

$$S=S_0\subset S_1\subset\cdots\subset S_m\subset\cdots\subset A\subset\cdots\subset B_m\subset\cdots\subset B_1\subset B_0=B,$$

then

$$(***)\quad \lambda_n^2(B)=\lambda_n^2(B_0)\leq\lambda_n^2(B_1)\leq\cdots\leq\lambda_n^2(B_m)\leq\cdots\leq\lambda_n^2(A)\leq\cdots\leq\lambda_n^2(S_m)\leq\cdots\leq\lambda_n^2(S_1)\leq\lambda_n^2(S_0)=\lambda_n^2(S).$$

Also

$$0\leq\lambda_1^2(B_m)\leq\lambda_2^2(B_m)\leq\lambda_3^2(B_m)\leq\cdots\ (m=1, 2, 3, \cdots),$$

and

$$0\leq\lambda_1^2(S_m)\leq\lambda_2^2(S_m)\leq\lambda_3^2(S_m)\leq\cdots.$$

In the ordinary Rayleigh-Ritz method one chooses first the sequence of linearly independent admissible functions $\psi_1, \psi_2, \psi_3, \cdots$, the "coordinate" functions of Ritz. For each $m=1, 2, 3, \cdots$, the set B_m consists of all linear combinations with constant coefficients of the first m "coordinate" functions $\psi_1, \psi_2, \cdots, \psi_m$, and (***) yields $\lambda_n^2(A)\leq\lambda_n^2(B_m)$ $(n=1, \cdots, m)$, $\lambda_n^2(A)\leq\infty$ $(n=m+1, m+2, \cdots)$, that is, finite upper bounds for only the first m eigenvalues $\lambda_n^2=\lambda_n^2(A)$ $(n=1, \cdots, m)$. In the generalized Rayleigh-Ritz method devised by Aronszajn, the initial set S_0, smaller than A, may be an infinite-dimensional subset of A, so that one can obtain in this way finite upper bounds for all the $\lambda_n^2=\lambda_n^2(A)$ at the first step, and hence at every step. In the Weinstein method, the initial set of functions B_0, bigger than A, is the set of arbitrary functions w satisfying only the boundary condition $w=0$ on C. In Weinstein's terminology this is described as "cancelling" the boundary condition $\partial w/\partial n=0$ in order to obtain the "base problem" P_0. Now let $p_1(s)$, $p_2(s)$, $\cdots$, $p_k(s)$, $\cdots$, where s denotes arc length on C, be a sequence of real-valued functions assumed to be complete on C in the sense that $w=\partial w/\partial n=0$ on C is equivalent to

$$\int_C p_k(s)\frac{\partial w}{\partial n}ds=0$$

for every $k=1, 2, \cdots$. For each $m=1, 2, \cdots$, the set of functions B_m, bigger than the set A, is taken by Weinstein to be defined by the boundary conditions $w=0$, $\int_C p_k(s)(\partial w/\partial n)ds=0$, for $k=1, 2, \cdots, m$. The variational problems obtained from (**) by replacing B_m by A, for $m=1, 2, 3, \cdots$, are called "intermediate problems" by Weinstein, and it is shown that, for every fixed n and increasing m, the nth eigenvalue of the mth intermediate problem converges to the nth eigenvalue of the original problem. The exact multiplicity of the eigenvalues of the mth intermediate problem is obtained by Aronszajn by consideration of the zeros and poles (in Weinstein's original work, of the zeros only, but with some additional considerations) of a certain meromorphic function $W_m(\zeta)$ called Weinstein's determinant.

The following is a summary of the contents of the book. Chapters 1 and 2 deal with vibrating systems that have a finite number of degrees of freedom and with the corresponding finite-dimensional variational principles. One

finds here a discussion of normal coordinates, normal modes of vibration, the recursive variational characterization of the eigenvalues of a finite-dimensional self-adjoint, positive definite operator, the maximum-minimum characterization of these eigenvalues, Rayleigh's comparison theorems for eigenvalues of systems vibrating under linear constraints (Section 8 of Chapter 2 gives a concise comparison of the methods of Rayleigh-Ritz and Weinstein). Chapters 3 and 6 present, respectively, the methods of Rayleigh-Ritz and Weinstein in their original form, for vibrating systems with infinitely many degrees of freedom; in particular, strings, rods, membranes and plates. In Section 12 of Chapter 3 is found a clear explanation of the ordinary Rayleigh-Ritz method for numerical calculation of the eigenvalues of a positive definite differential operator, together with a numerical example. Chapter 6 considers the Weinstein method in its original form (as in the Mémorial volume quoted above), without reference to Hilbert space or the theory of operators (questions of existence and convergence being left to later chapters), so that this chapter can be read independently. The results of applying this method to the computation of lower bounds for the first thirteen eigenvalues of a clamped square plate are tabulated on page 154, from a mimeographed report of N. Aronszajn [Application of Weinstein's method with an auxiliary problem of type I, Univ. of Kansas]. Chapter 8 describes and justifies the actual details of the Weinstein-Aronszajn method for a completely continuous operator in abstract Hilbert space [Aronszajn, Operators in a Hilbert space, Univ. of Kansas], where existence and convergence are readily proved. Chapter 9 carries out the task of transforming a given eigenvalue problem, in particular, the vibrating plate problem, into an eigenvalue problem for a linear operator in Hilbert space (in this way answering the questions of existence and convergence, on the basis of concepts introduced in Chapters 4, 5, and 7; the latter are devoted, respectively, to reproducing kernels and functional completion [Aronszajn, Trans. Amer. Math. Soc. **68** (1950), 337–404; MR **14**, 479], Green's functions for problems of vibrating rods, membranes and plates, and linear operators in a Hilbert space. Finally, Chapter 10 deals with the extensions, by Aronszajn and his collaborators, to more general differential problems.

The printing and the format of the book are excellent. The volume reads very smoothly; the author's expository skill is certainly of first magnitude. The exercises are carefully designed to throw light on the material expounded, and add greatly to the attractiveness of this work as a text. Many a scientist will be thankful for the author's labors in providing such a remarkably easy access to the theory and practice of these variational methods for calculating eigenvalues. *J. B. Diaz.*

Il'in, V. A. Theorem on the possibility of expansion of a piecewise smooth function according to eigenfunctions of an arbitrary two-dimensional region. Dokl. Akad. Nauk SSSR (N.S.) **109** (1956), 442–445. (Russian)

The author considers boundary value problems associated with the Laplacian and certain homogeneous boundary conditions on a two-dimensional region G. It is shown that any piecewise-smooth function which satisfies the appropriate boundary conditions can be expanded in a series of eigenfunctions, and the series (if summed in order of increasing eigenvalues) converges uniformly on any interior subdomain not containing neighborhoods of contours on which the function has discontinuities. A piecewise-smooth function is one which is piecewise-continuous, and has piecewise-continuous first and second derivatives. A piecewise-continuous function f on G is one which is continuous in G except possibly on a finite number of sufficiently smooth contours inside G, where f has discontinuities of the first kind. The proof reduces the problem to the solved case where f is continuous, but with piecewise-continuous first and second derivatives. *E. A. Coddington* (Princeton, N.J.).

See also: Harmonic Functions, Convex Functions: Magenes. **Special Functions:** Maass. **Ordinary Differential Equations:** Zubov; Kato; Ladyženskaya. **Topological Vector Spaces:** Gel'fand. **Banach Spaces, Banach Algebras, Hilbert Spaces:** Altman (fourth article). **Probability:** Hille. **Elasticity, Plasticity:** Ryabova. **Optics, Electromagnetic Theory, Circuits:** Wilcox.

Difference Equations, Functional Equations

Evangelisti, Giuseppe. Il calcolo delle differenze finite nella matematica applicata. Rend. Sem. Mat. Fis. Milano **26** (1954–55), 69–87 (1957).

A summary account of the applications of finite differences to interpolation, quadrature and the numerical solution of differential equations. Difference equations are set up for the problem of three moments, the hydraulic ram, and a cascade of electric cells.

L. M. Milne-Thomson (Providence, R.I.).

Berezanskiĭ, Yu. M. Expansion according to eigenfunctions of a partial difference equation of order two. Trudy Moskov. Mat. Obšč. **5** (1956), 203–268. (Russian)

This article elaborates proofs for a number of the author's theorems [Dokl. Akad. Nauk SSSR (N.S.) **93** (1953), 5–8; **97** (1954), 573–576; MR **16**, 713]. In addition, an inverse problem of spectral analysis is considered, the construction of the coefficients of the difference equation

$$Lu_{j,k}=a_{j-1,k}u_{j-1,k}+a_{j,k}u_{j+1,k}+b_{j,k-1}u_{j,k-1}+b_{j,k}u_{j,k+1}+c_{j,k}u_{j,k}=\lambda u_{j,k}$$

from their spectral matrix. Some illustrative examples are given. *U. W. Hochstrasser* (Lawrence, Kans.).

Ålander, Martin. Sur les solutions périodiques d'une certaine équation fonctionnelle. C. R. Acad. Sci. Paris **244** (1957), 2118–2120.

The author determines the periodic entire solutions of the equations (1) $f'(z)=f(z+\alpha)$, (2) $f'(z)=af(z+\beta)$, where α is real; (2) can be reduced to (1). The solutions of (1) are certain exponentials or linear combinations of two exponentials, which are described in detail.

R. P. Boas, Jr. (Evanston, Ill.).

Osserman, Robert. On the solution of $f(f(z))=e^z-1$ and its domain of regularity. Proc. Amer. Math. Soc. **8** (1957), 262–263.

Let S denote the strip $|I(z)|<\pi$ and $\bar{S}$ the closure of S. The following result is established: If $f(z)$ is defined in some open region $D\supset\bar{S}$, if $f(\bar{S})\subset D$, and if $f(z)$ satisfies the equation (*) $f(f(z))=e^z-1$ throughout D, then $f(z)$ cannot be analytic in D. From the last part of the proof a reader might draw the inference that a solution of (*) holomorphic in the strip S and with a singularity at $\log(c-1)+i\pi$ actually exists. This, however, is not what is proved in this article. It is quite possible that a solution of (*) holomorphic in S does not exist at all. *W. J. Thron.*

See also: Programming, Resource Allocation, Games: Duffin.

Integral and Integrodifferential Equations

See: Ordinary Differential Equations: Jost. **Partial Differential Equations:** Kolodner. **Banach Spaces, Banach Algebras, Hilbert Spaces:** Birkhoff; Altman (four articles). **Elasticity, Plasticity:** Nowacki and Sokołowski.

Calculus of Variations

Brunk, H. D.; Ewing, G. M.; and Utz, W. R. Minimizing integrals in certain classes of monotone functions. Pacific J. Math. **7** (1957), 833–847.

Let R_n be the space of points $t=(t^1, \cdots, t^n)$, μ a totally finite, complete measure on R_n, and $\alpha(t)$ a measurable real-valued function. Let F be a real-valued function on R_2 satisfying: (1) $F[\alpha(t), \theta(t)]$ is measurable if $\theta(t)$ is; (2) for fixed u, $F(u, v)$ is strictly decreasing (increasing) in v for $v \leqq u$ ($v \geqq u$) and right (left) continuous in v for $v<u$ ($v>u$);

(3) $\int F[\alpha(t), \alpha(t)]d\mu(t)<\infty$. Let $J(\theta)=\int F[\alpha(t), \theta(t)]d\mu(t)$.

The authors show the existence of functions minimizing $J(\theta)$ for θ in two classes of monotone functions. One class consists of those $\theta(t)$ which are non-decreasing in each component of t and for which all of the second differences, obtained from pairs of components, are negative. Representation and uniqueness theorems are also proved. The problem is connected with that of obtaining a maximum likelihood estimate of a distribution function.

E. Silverman (East Lansing, Mich.).

See also: Measure, Integration: Cesari. **Partial Differential Equations:** Gould. **Geometries, Euclidean and Other:** Busemann.

TOPOLOGICAL ALGEBRAIC STRUCTURES

Topological Groups

Freudenthal, Hans. Neuere Fassungen des Riemann-Helmholtz-Lieschen Raumproblems. Schr. Forschungsinst. Math. **1** (1957), 92–97.

Various solutions of the Riemann-Helmholtz problem are first reviewed in historical order. Following a program initiated by A. Kolmogoroff [Nachr. Ges. Wiss. Göttingen. Math.-Phys. Kl. **1930**, 208–210], and following indications [Bull. Soc. Math. Belgique **1952**, 44–52 (1953); MR **15**, 334] and an unpublished manuscript of J. Tits, new postulates for free mobility are then given in terms of the group G of "rigid motions" of the space R considered. (The meaning of these postulates is most easily seen if R is assumed to be a metric space, and G the group of all isometries of R.) The three postulates are: (S) if A is closed in R, and B compact in R and disjoint from A, then R contains a non-void open set U so small that, for no $\gamma \in G$ can $U\gamma$ overlap both A and B; (V) G is complete; (Z) for some $a \in R$, the "sphere" of all $a\beta$ (β in the subgroup $B \leqq G$ of isotopy of some $b \neq a$) separates R. It is asserted that, using Yamabe's solution of the generalized Fifth Problem of Hilbert [Ann. of Math. (2) **58** (1953), 351–365; MR **15**, 398], G can be shown to be a closed subgroup of the Euclidean, elliptic, spherical or hyperbolic group with real, complex, real quaternion, or Cayley number coefficients. As a corollary, let R be any locally compact and connected metric space, and let the group G of isometries of R be transitive on all point-pairs separated by a distance $\delta>0$. Then G is one of the above subgroups. If G is transitive on congruent point-triples, then the field in question is the real field.

G. Birkhoff.

Tits, J. Transitivité des groupes de mouvements. Schr. Forschungsinst. Math. **1** (1957), 98–111.

Verf. berichtet an Hand von 32 Literaturstellen über einige neue Resultate bezüglich des Riemann-Helmholtz-Lie'schen Problems und seiner Verallgemeinerung, d.h., die Charakerisierutng der Bewegungsgruppe einer Klasse von Räumen durch Transitivitätseigenschaften der Gruppe, ein Problem, zu dem zahlreiche Arbeiten des Verf. einen Beitrag liefern. Insbesondere bestätigt Verf. hier eine früher von ihm ausgesprochene Vermutung [Acad. Roy. Belg. Mém. Coll. in 8° **29** (1955), no. 3; C. R. Acad. Sci. Paris **239** (1954), 526–527; MR **17**, 874; **16**, 219] bezüglich der Ausnahmefälle seiner dort angegebenen Klassifikation, und er kann jetzt die zweifach-transitiven stetigen Transformationsgruppen unter ziemlich allgemeinen topologischen Voraussetzungen vollständig klassifizieren, woraus sich dann leicht eine Übersicht über alle stetigen n-fach transitiven ($n \geqq 3$) Gruppen gewinnen lässt. Das Studium der beliebigen n-fach transitiven Gruppen bietet im Gegensatz zu den im engeren Sinn n-fach transitiven Gruppen (d.h., es gibt nur eine Transformation, die n gegebene Punkte in n gegebene Punkte überführt) erhebliche, noch ungelöste Schwierigkeiten. Verf. klassifiziert ferner die stetige Automorphismengruppen einer differenzierbaren Mannigfaltigkeit, die transitiv auf den Tangentenrichtungen der Mannigfaltigkeit operiert, d.h., transitiv auf den Paaren unendlich benachbarter Punkte.

R. Moufang (Frankfurt am Main).

Taylor, Robert L. Covering groups of nonconnected topological groups. Proc. Amer. Math. Soc. **5** (1954), 753–768.

Eine Ueberlagerung der topologischen Gruppe G wird definiert als ein Paar (E, φ) bestehend aus einer topologischen Gruppe E und einem Homomorphismus φ von E auf G, welcher ein lokaler Isomorphismus ist; oder äquivalent, welcher eine Faserabbildung ist (Existenz lokaler Schnittflächen) und dessen Kern $\varphi^{-1}(e)$ diskret ist. Die Ueberlagerung (E, φ) von G heisst eigentlich, wenn das Urbild $\varphi^{-1}(G_e)$ der Neutralkomponente G_e von G zusammenhängend ist. Während in der Ueberlagerungstheorie von Räumen nur der zusammenhängende Fall von Interesse ist, ergeben sich für eigentliche Ueberlagerungen nicht-zusammenhängender Gruppen neue Gesichtspunkte; in der vorliegenden Arbeit wird, soweit möglich, die für zusammenhängende Räume bekannte Beziehung zwischen Fundamentalgruppe und Ueberlagerung auf diese Situation ausgedehnt.

(G_e^*, p) sei die universelle Ueberlagerung von G_e im üblichen Sinne, $p^{-1}(e)=\pi$ die Fundamentalgruppe von G_e, (E, φ) eine eigentliche Ueberlagerung von G. Es gibt einen bestimmten Homomorphismus W von G_e^* in E mit $p=\varphi W$. Die Untergruppe $W^{-1}(e)$ von π heisst der Typus von (E, φ). Bei vorgegebener Untergruppe ρ von π braucht nicht immer eine eigentliche Ueberlagerung vom Typus ρ zu existieren, oder es kann mehrere nicht-isomorphe geben. Ist (K, q) die zu $\rho \subset \pi$ gehörende Ueberlagerung von

G_e (d.h. $K=G_e^*/\rho$, q durch p induziert), so besteht eine eineindeutige Zuordnung zwischen den Klassen isomorpher eigentlicher Ueberlagerungen vom Typus ρ und den Klassen isomorpher „Fortsetzungen" von q (Homomorphismen f von topologischen Gruppen, die K enthalten, auf G, mit $f|K=q$ und $f^{-1}(e)=q^{-1}(e)$). Die Existenzbedingung für eigentliche Ueberlagerungen von vorgegebenem Typus ρ und Aussagen über deren Anzahl werden in der Sprache der Cohomologietheorie der abstrakten Gruppen formuliert und in speziellen Beispielen explizit untersucht. Insbesondere wird der Fall der drei 2-komponentigen 1-dimensionalen kompakten Lieschen Gruppen behandelt: die Anzahl der eigentlichen, einfach zusammenhängenden Ueberlagerungen beträgt für die eine 1, für die zweite 0 und für die dritte 2.

B. Eckmann (Zürich).

Leptin, Horst. Eine Kennzeichnung der reinen Untergruppen abelscher Gruppen. Acta Math. Acad. Sci. Hungar. **7** (1956), 169–171. (Russian summary)

Theorem: A closed pure subgroup of a compact abelian group is (algebraically) a direct summand. The proof uses the structure of compact abelian groups and an analogous theorem on linearly compact modules which appears in Math. Z. **66** (1957), 289–327 [MR **19**, 245].

I. Kaplansky (Princeton, N.J.).

Ellis, J. W. Duality in products of groups with operators. Trans. Amer. Math. Soc. **83** (1956), 301–312.

Duality theorems for products of locally convex topological spaces [cf. M. Katětov, Acta Fac. Nat. Univ. Carol., Prague no. **181** (1948); MR **10**, 127] and of topological groups [cf. S. Kaplan, Duke Math. J. **15** (1948), 649–658; MR **10**, 233] are generalized. Instead of groups or linear spaces, "S-spaces" are considered (S is a commutative semigroup; an S- space is an abelian group upon which $s\in S$ act as homomorphisms). If X, Y, Z are S-spaces, Y, Z are topologized, then X is called a Z-complement of Y if there is an S-homomorphism on X onto the set of all continuous S-homomorphisms on Y into Z; X, Y are called Z-dual if they are Z-complements of each other. Let $\mathfrak{B}$ be an additive ideal of subsets of a set A; $\mathfrak{B}'$ will denote the ideal of all $E\subset A$ such that every $E\cap B$, $B\in\mathfrak{B}$, is finite. If X_a, $a\in A$, are S-spaces, a product $(\mathfrak{B}P)X_a$ (reviewer's notation) consists of all $x=\{x_a\}\in PX_a$ (cartesian product) such that $\{a: x_a\neq 0\}\in\mathfrak{B}$. Main theorems of the note assert that, under certain restrictions (e.g., it is required that the topology of Y should lie within a certain interval), if X_a are Z-complements of Y_a, then $X=(\mathfrak{B}'P)X_a$ is a Z-complement of $Y=(\mathfrak{B}P)Y_a$; if $\mathfrak{B}''=\mathfrak{B}$ and X_a, Y_a are Z-dual, then X and Y are Z-dual. Kaplan's and the reviewer's duality theorems as well as various other results follow as special cases. A result of J. Dixmier [Ann. of Math. (2) **51** (1950), 387–408; MR **11**, 441] concerning linear functionals on the space of bounded operators in a Hilbert space appears as a corollary.

M. Katětov (Prague).

Reiter, H. Contributions to harmonic analysis. Acta Math. **96** (1956), 253–263.

If φ is a bounded measurable function on a locally compact abelian group G, the spectrum of φ is defined to be the set of continuous characters lying in the weakly-closed (in the sense of $L^\infty=(L^1)^*$) translation-invariant subspace generated by φ. Let g be a closed subgroup of G, and Γ the corresponding subgroup of the character group $\hat{G}$; that is, $\hat{g}=\hat{G}/\Gamma$. Thus, if $x\in G$ is fixed, the spectrum of $\varphi(xs)$, as a function of $s\in g$, is a subset of $\hat{G}/\Gamma$. The author proves that if φ is continuous then this subset is always included in the closure of the image of the spectrum of φ in $\hat{G}/\Gamma$. This theorem is then applied to deduce information about the spectra of functions φ of various special types, and, consequently, to the explicit calculation of the norm in a quotient algebra $L^1(G)/I$ for a certain type of closed ideal I.

L. H. Loomis (Cambridge, Mass.).

Numakura, Katsumi. Theorems on compact totally disconnected semigroups and lattices. Proc. Amer. Math. Soc. **8** (1957), 623–626.

A mob is a Hausdorff topological semigroup. The author proves that a compact totally disconnected mob is a projective limit of finite discrete mobs, and then shows how his methods may be extended to demonstrate analogous statements about other topological algebraic systems. An imbedding theorem for compact totally disconnected lattices is proved.

H. H. Corson.

Koch, R. J. On monothetic semigroups. Proc. Amer. Math. Soc. **8** (1957), 397–401.

[For background, terminology, and related results, see A. H. Clifford, Amer. J. Math. **70** (1948), 521–526; MR **10**, 12; R. Ellis, Proc. Amer. Math. Soc. **4** (1953), 969–973; MR **15**, 729; W. H. Gottschalk, Ann. of Math. (2) **47** (1946), 762–766; MR **8**, 159; J. L. Kelley, Duke Math. J. **17** (1950), 277–283; MR **12**, 194; R. J. Koch, Proc. Amer. Math. Soc. **5** (1954), 828–833; MR **16**, 447; K. Numakura, Math. J. Okayama Univ. **1** (1952), 99–108; Bull. Yamagata Univ. (Nat. Sci.) **1951**, no. 4, 405–411; MR **14**, 18; **16**, 447; J. E. L. Peck, Proc. Amer. Math. Soc. **2** (1951), 414–421; MR **12**, 835; A. D. Wallace, An. Acad. Brasil. Ci. **24** (1952), 329–334; Math. J. Okayama Univ. **3**(1953), 23–28; MR **14**, 724; **15**, 933.] A semigroup (sg) is a Hausdorff space S with an associative binary and bi-continuous operation. S is a monothetic sg (msg) if the set $\Gamma(a)$ of all positive powers of some $a\in S$ has S as its closure; a is called a generator of S. For compact msg's S it is shown that S is a topological group if either there exists a nonisolated generator or if there exist two distinct generators. The proof is based on the following result due independently to Numakura, Peck and the author (see bibliography at beginning of this review): A compact msg S with generator a has the property that the cluster points of the net $\{a^n\}$ form a group which is the minimal ideal of S. After some applications to sg's acting on topological spaces which yield simplifications of known results of Numakura and Wallace, the author turns to an analog of Gottschalk's extension of a theorem due to Weil on monothetic (in the conventional sense) groups: a locally compact msg S with kernel is compact; a locally compact msg S with unit, whose component is compact, is a compact group.

G. K. Kalisch.

See also: Algebras: Ree. **Integral Transforms:** Hewitt and Wigner. **Algebraic Geometry:** Lang.

Lie Groups and Algebras

Dieudonné, Jean. On the Artin-Hasse exponential series. Proc. Amer. Math. Soc. **8** (1957), 210–214.

In his work on formal Lie groups, Dieudonné determined all homomorphisms of the additive group of Witt vectors, over a field of prime characteristic, into the multiplicative group of power series over this field, that

can be described by formal power series [Mathematika 2 (1955), 21–31; MR 17, 175]. A special homomorphism of this type was defined by Whaples [Duke Math. J. 21 (1954), 575–581; MR 17, 464], using a series introduced by Artin and Hasse. {Whaples merely replaced the coefficients of the Artin-Hasse series by their residue classes mod p.} In this paper Dieudonné determines the connection between his homomorphisms and those of Artin-Hasse. *G. Whaples* (Bloomington, Ind.).

Bott, Raoul. An application of the Morse theory to the topology of Lie-groups. Bull. Soc. Math. France **84** (1956), 251–281.

Let G be a compact, connected, simply connected Lie group. Then G may be given a Riemannian structure invariant under left and right translations. The author applies Morse's theory to this Riemannian manifold in order to obtain topological results relative to G. The geodesics of G are the translated curves of the one-parameter subgroups. On the other hand, G may be made to act upon itself by associating to every $\sigma \in G$ the mapping $\tau \to \sigma\tau\sigma^{-1}$; the elements X of the Lie algebra $\mathfrak{g}$ of G thus determine vector fields $\pi(X)$ on G. Let N be the orbit of a point of G and g a geodesic starting normally to N from a point of N. In the Morse theory, one has to consider the "J-fields" of tangent vectors to G along g; roughly speaking, a J-field is the vector field along g determined by a geodesic infinitely near to g and also starting normally to N at a point of N. If $X \in \mathfrak{g}$, the restriction of $\pi(X)$ to g is a J-field; although not every J-field is of this form, nevetheless those which occur in the Morse theory, i.e., those which vanish at some point of g, are actually of the form $\pi(X)$; the author expresses this fact by saying that the action of G upon itself is variationally complete. Let σ_1 be a regular point of G (i.e., its orbit is of maximal dimension) not on N; then σ_1 is also regular in the sense of Morse theory (i.e., if g goes through σ_1, no J-field $\neq 0$ along g may vanish at σ_1). Assuming that g goes through σ_1, let s be the segment of g between its starting point on N and σ_1. Then Morse's theory involves the number $\lambda(s)$=sum for all $\tau \in s$ of the dimensions of the spaces $\Lambda(\tau)$, $\Lambda(\tau)$ being the set of J-fields which vanish at τ. This number may be computed and turns out to be the sum for all $\tau \in s$ of the numbers dim $o(\sigma_1)$—dim $o(\tau)$, where $o(\sigma)$ is the orbit of σ. On the other hand, g lies in the maximal torus T which contains σ_1, and s may be represented by a straight line segment $\tilde{s}$ in the Lie algebra $\mathfrak{t}$ of T. The points $X \in \mathfrak{t}$, such that $\exp X$ is singular, form a family Φ of hyperplanes of $\mathfrak{t}$, and $\lambda(s)$ turns out to be twice the number of hyperplanes in Φ which are crossed by $\tilde{s}$. The fact that these numbers are all even is precisely the fact which allows inference of precise topological results from the knowledge of the numbers $\lambda(s)$, because it then follows from Morse's inequalities that $\sum_s t^{\lambda(s)}$ (the summation being extended to all s which end at σ_1 and start normally to N at points of N) is equal to the Poincaré series of the space $\Omega(G, N)$ consisting of all paths of G leading from points of N to σ_1. This being true for any basic field, it follows that $\Omega(G, N)$ has no torsion; its odd-dimensional Betti numbers are clearly 0. Specializing to the case where $N=\{\varepsilon\}$ (ε=the neutral element), it is seen that the space $\Omega(G)$ of loops of G has no torsion and that its odd-dimensional Betti numbers are 0. Moreover, explicit expressions are given for the Poincaré series of $\Omega(G)$ and $\Omega(G, N)$.

Next, the author introduces an open neighbourhood M of ε which is the image under the exponential mapping of a solid sphere of center 0 in $\mathfrak{g}$ (in the euclidean metric which is invariant under the adjoint representation). Then G still acts on M in the same manner as above; assuming that $N \subset M$, $\sigma_1 \in M$, the author obtains results similar to those previously described but bearing now on the space $\Omega(M, N)$ of paths leading in M from points of N to σ_1. However, this last space is of the same homotopy type as N itself, as easily seen. In this manner, the author proves that, if C is the centralizer of any torus in G, then G/C has no torsion, and he gives an explicit expression for the Poincaré polynomial of G/C.

In an annex, it is shown that the results of the paper may be used to derive anew the values of the Betti numbers of the groups E_6, E_7, E_8. *C. Chevalley.*

Montgomery, D.; and Yang, C. T. The existence of a slice. Ann. of Math. (2) **65** (1957), 108–116.

Let G be a compact Lie group acting on a space M (separable metric, complete, finite-dimensional). The authors show that at each point p of M there exists a "slice" (generalized cross-section), i.e., a closed set K such that (1) $p \in K$, (2) $G_p(K)=K$, (3) $y \in K \Rightarrow G_y \subset G_p$ and $G_p(y)=K \cap G(y)$, (4) for a small cell R in G, transversal to G_p at e, the map $R \times K \to R(K)$ is topological and $R(K)$ is a neighborhood of p. Here G_y is the usual stability group of y in G. The proof is based on the study of appropriate decomposition spaces, Gleason's theorem on cross-sections [Proc. Amer. Math. Soc. **1** (1950), 35–43; MR 11, 497], and Michael's theorem on selections [Ann. of Math. (2) **64** (1956), 562–580; MR **18**, 325]. It is also shown that the orbit space M^* is finite-dimensional. Application: an arc in M^* can always be lifted into M. A similar result is proved in the paper reviewed below. *H. Samelson* (Ann Arbor, Mich.).

Mostow, G. D. Equivariant embeddings in Euclidean space. Ann. of Math. (2) **65** (1957), 432–446.

Let E be a separable, metric, finite-dimensional space, and G a compact Lie group of transformations acting effectively on E. An equivariant imbedding in Euclidean space is a homeomorphism φ of E into some E^n and an isomorphism θ of G into the unitary group on E^n such that $\varphi(g(x))=\theta(g)\varphi(x)$, $x \in E$, $g \in G$. The map φ is said to be n.t. if the origin is the only fixed point of $\theta(G)$. The principal result states that a necessary and sufficient condition for $\{E, G\}$ to have an equivariant imbedding is that there exist just a finite number of inequivalent orbits of G in E (i.e., there exist a finite number of subgroups H_i such that, for $x \in E$, the isotropy group G_x is conjugate to one of the H_i's). Moreover, if G has no fixed points in E, then the imbedding can be chosen n.t.

Several results of interest are proved in the process of proving the main result. The author defines a pseudo-section [='slice' of Montgomery, Samelson and Yang, Ann. of Math. (2) **64** (1956), 131–141; MR **17**, 1224] to the orbit $G(p)$ at p as a closed set K containing p such that: (1) K is invariant under G_p; (2) there exists a continuous cross-section map f into G of a neighborhood U of the coset G_p in G/G_p such that the mapping $(u, x) \to f(u)(x)$ is a homeomorphism of the product space $U \times K$ onto a neighborhood of p; (3) $g(K) \cap K$ is empty if $g \in G-G_p$. The author proves that if E is completely regular and G is a compact Lie group of transformations on E, then a pseudo-section exists at every point of E. This is a generalization, in a highly useful form, of results of Koszul [Géométrie différentielle, Colloq. Internat. Centre Nat. Rech. Sci., Strasbourg, 1953, pp. 137–141; MR **15**, 600],

Montgomery, Samelson, and Yang [loc. cit.] and Montgomery and Yang [see the paper reviewed above].

Another interesting result is as follows: let G be a compact Lie group of transformations on a separable, metric, finite-dimensional space E. If all the orbits are equivalent, then there exists a finite set of local cross-sections whose orbits cover E. An important tool in the proof of this theorem, and in turn in other proofs, is the following purely topological theorem: Let E be a finite-dimensional, separable, metric space; then any covering of E has a star bounded refinement. Finally, a theorem on compact Lie groups is proved: In a compact Lie group, any set of (connected) analytic subgroups whose normalizers are mutually non-conjugate is finite. Any set of semi-simple analytic subgroups which are mutually non-conjugate is finite.

There are several other results of interest. The techniques should become standard equipment for the worker in this area. *P. S. Mostert* (New Orleans, La.).

Floyd, E. E. Orbits of torus groups operating on manifolds. Ann. of Math. (2) **65** (1957), 505–512.

Let G be a compact connected Lie group operating on a compact manifold X. If $x \in X$, G_x denotes the isotropy subgroup of G at x. Montgomery asks whether there are only a finite number of conjugate classes of the G_x's. In this paper the author answers the equation in the affirmative for the case that G is a torus group. This result leads to an affirmative answer in the general case by Mostow [see the paper reviewed below]. Instead of ordinary manifolds the author deals with homology manifolds mod p for all prime p. The basic procedure is to use the existence in G (a torus group) of a plentiful supply of subgroups of prime-power order and then to apply the Smith special homology theory. The key theorem is: If $\{G_i\}$ is a sequence of prime-power subgroups of G convergent to G, the fixed-point set of G_i is equal to the fixed-point set of G when i is sufficiently large.

C. T. Yang (Philadelphia, Pa.).

Mostow, G. D. On a conjecture of Montgomery. Ann. of Math. (2) **65** (1957), 513–516.

If G is a compact Lie group operating on a space M, then the isotropy subgroup G_p, $p \in M$, is the set of all g with $gp=p$. In answer to a well-known conjecture of Montgomery, it is here proved that if the compact Lie group G operates on a compact manifold M, then there are only a finite number of non-conjugate isotropy subgroups. Equivalently, there are only a finite number of equivalence classes of orbits. Use is made of the recent theorem of the reviewer [see the paper reviewed above] that this theorem holds in case G is a toroid and M is a compact orientable manifold. A first step is to remove the orientability hypothesis in that result. It is then shown that (1) there are only a finite number of normal subgroups of the form G_p^e, where G_p^e is the identity component of G_p. This portion uses the structure theory for Lie groups as well as the result concerning toroids. It then follows easily that (2) no more than a finite number of the G_p^e have the same normalizer. By a recent result of the author [see the paper reviewed second above], there are at most a finite number of non-conjugate subgroups of G which are normalizers of connected analytic subgroups. Hence (3) there are at most a finite number of non-conjugate subgroups of the form G_p^e. Finally, it is shown that (4) there are at most a finite number of G_p with a prescribed G_p^e. Again, use is made of the structure theory of Lie groups and the result concerning toroids.

E. E. Floyd (Charlottesville, Va.).

Yang, Chung-Tao. On a problem of Montgomery. Proc. Amer. Math. Soc. **8** (1957), 255–257.

The author proves the following theorem. Let G be a compact Lie group which acts differentiably on a manifold M; then each point $p \in M$ has a neighborhood U, such that the stability groups G_y, for $y \in U$, belong to a finite number of conjugate classes. The proof depends on the existence of a slice [cf. Montgomery, Samelson, and Yang, Ann. of Math. (2) **64** (1956), 131–141; MR **17**, 1224]. A counter example is provided for the non-compact case. It might be noted that differentiability has been removed from the hypothesis by G. D. Mostow [see the paper reviewed above]. *H. Samelson* (Ann Arbor, Mich.).

Berezin, F. A. Laplace operators on semisimple Lie groups and on certain symmetric spaces. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 1(73), 152–156. (Russian)

The author defines a Laplace operator and radial part of a Laplace operator as in his earlier note on the same subject [Dokl. Akad. Nauk SSSR (N.S.) **107** (1956), 9–12; MR **17**, 1109]. Then he states a number of theorems concerning the exact form of such operators in various cases. Some of these are special cases of theorems announced in his earlier note. *G. W. Mackey* (Cambridge, Mass.).

Berezin, F. A. Representation of complex semisimple Lie groups in Banach space. Dokl. Akad. Nauk SSSR (N.S.) **110** (1956), 897–900. (Russian)

The paper lists all irreducible representations (up to equivalence) of complex semi-simple Lie groups in Banach spaces. The method is based on the following theory. With any representation there is associated a representation of the ring of Laplace operators on the group (i.e., the centre of the Birkhoff-Witt algebra interpreted as the algebra of left invariant differential operators on the group); if the group representation is irreducible, the associated representation for the Laplace ring is a representation by scalar multiplications, and therefore essentially a homomorphism of this ring into the complex numbers. To a given homomorphism λ of the Laplace ring into the complex numbers there correspond at most a finite number of non-equivalent representations, since the characters of these representations are the solutions of a certain eigenvalue problem depending on λ with at most a finite number of linearly independent solutions. Finally, it turns out that for any homomorphism λ a fundamental set of solutions for the associated eigenvalue problem is provided by the characters of so-called elementary representations, which are exhibited explicitly, whence a description of the irreducible representations in Banach spaces is obtained. At the same time it follows that any unitary irreducible representation must be equivalent to one of the representations exhibited by Gelfand and Neumark [Trudy Mat. Inst. Steklov. **36** (1950); MR **13**, 722].

W. T. van Est (Leiden).

Harish-Chandra. Spherical functions on a semisimple Lie group. Proc. Nat. Acad. Sci. U.S.A. **43** (1957), 408–409.

Let G be a connected semi-simple Lie group with a

finite center, K a maximal compact subgroup of G, and $\mathfrak{g}_0$ the Lie algebra of G over the field of real numbers. This paper is concerned with the problem of determining explicitly the Plancherel formula for G/K. Certain elementary spherical functions and integral expressions in terms of them are considered. In particular an explicit expression for the square-integral of a spherical function is obtained (Corollary 1 to Theorem 3) and it is stated that this would imply an explicit Plancherel-formula for G/K, provided it could be shown that a certain class of functions is dense in the space of all square integrable spherical functions.

F. I. Mautner (Baltimore, Md.).

Harish-Chandra. Fourier transforms on a semisimple Lie algebra. I. Amer. J. Math. **79** (1957), 193–257.

In this paper certain integral formulas on semisimple Lie groups and Lie algebras are studied, which are closely related to the problem of computing explicitly the Plancherel formula on any semisimple Lie group. Let $\mathfrak{g}_0$ be a real semisimple Lie algebra and $\mathfrak{h}_0$ a Cartan subalgebra of $\mathfrak{g}_0$. Let G be the connected adjoint group of $\mathfrak{g}_0$ and A the Cartan subgroup of G corresponding to $\mathfrak{h}_0$. Let $G^*=G/A$. It is shown that the integral

$$\int_{G^*} f(x^*Hx^{*-1})dx^* \quad (x^* \in G^*,\ H \in \mathfrak{h}_0)$$

is convergent for suitably restricted functions f defined on $\mathfrak{g}_0$. In terms of this integral $cf(0)$ can be expressed as a limit as $H \to 0$, where c is a constant independent of f. The author states that $\mathfrak{h}_0$ and a certain submanifold of $\mathfrak{h}_0$ can be chosen such that $c \neq 0$ when H tends to 0 in this submanifold, but that the proof of these statements is rather long and will be completed only in the next paper of this series. In the case of the real unimodular group the above mentioned expression for $f(0)$ has previously been obtained by Gelfand and Graev [Dokl. Akad. Nauk SSSR (N.S.) **92** (1953), 461–464; MR **15**, 683].

Let K be a maximal compact subgroup of G and $\mathfrak{k}_0$ the corresponding Lie algebra. The (crucial) case when $\mathfrak{h}_0 \subseteq \mathfrak{k}_0$ is treated first and the general case then treated in terms of it. In Section 7 the partial result is obtained that the above c is a constant on each connected component of the set of regular elements of $\mathfrak{g}_0$. The problem is thus reduced to showing that these constants are equal. It is promised that this will be carried out in another paper. In Section 8 the notion of a fundamental Cartan subalgebra is defined as follows. Let w be the Casimir polynomial. Put $l_-(\mathfrak{h}_0)=\sup(\dim \mathfrak{h}_1)$, where $\mathfrak{h}_1$ runs over all linear subspaces of $\mathfrak{h}_0$ on which w is negative definite. Let $l_-=\sup l_-(\mathfrak{h}_0)$, where $\mathfrak{h}_0$ runs over all Cartan subalgebras of $\mathfrak{g}_0$. Then $\mathfrak{h}_0$ is called fundamental if $l_-(\mathfrak{h}_0)=l_-$. It is obvious that if $\mathfrak{h}_0$ is fundamental then so is any conjugate in G of $\mathfrak{h}_0$. It is proved that any two fundamental Cartan subalgebras of $\mathfrak{g}_0$ are conjugate in G. The author shows that the above c is 0 if $\mathfrak{h}_0$ is not fundamental. Therefore it remains only to study the case when $\mathfrak{h}_0$ is fundamental. The rest of the paper is mainly devoted to proving that for certain fundamental $\mathfrak{h}_0$ the constant c is not 0. This is done by applying an unpublished formula of de Rham on the elementary solution of certain types of differential operators to the Casimir operator.

F. I. Mautner.

See also: Algebras: Cartier; Ree. **Topological Groups:** Taylor. **Complex Manifolds:** Matsushima.

Topological Vector Spaces

Deprit, André. Sous-espaces vectoriels d'un espace vectoriel topologique séparé. Acad. Roy. Belg. Bull. Cl. Sci. (5) **43** (1957), 106–113.

Let E be a real or complex, separated, locally convex topological vector space. Let $M_1 \subset M_2 \subset \cdots \subset M_m$ and $L_1 \supset L_2 \supset \cdots \supset L_n$ be proper linear subspaces of E such that $M_j \neq M_{j+1}$, $L_k \neq L_{k+1}$. If each M_j is of finite dimension and if, for each k, there exists a continuous projection of E onto L_k, then there exist $(m+1)(n+1)$ linearly independent linear subspaces L_k^j $(1 \leq j \leq m+1,\ 0 \leq k \leq n)$ of E such that M_j and L_k are the topological direct sums

$$M_j = L_0^j \oplus L_1^j \oplus \cdots \oplus L_n^j \oplus M_{j-1} \quad (1 \leq j \leq m+1),$$

$$L_k = L_k^1 \oplus L_k^2 \oplus \cdots \oplus L_k^{m+1} \oplus L_{k+1} \quad (0 \leq k \leq n),$$

where $M_0 = L_{n+1} = \{0\}$, $M_{m+1} = L_0 = E$. This is similar to the author's previous result [same Bull. (5) **42** (1956), 1012–1017; MR **18**, 659], where the L_k's are assumed to be closed linear subspaces of finite codimension. Both are analogous to a topology-free theorem of M. Hukuhara [J. Fac. Sci. Univ. Tokyo. Sect. I. **7** (1954), 129–192; MR **16**, 992]. The present paper also contains a similar theorem for topological vector spaces over a more general field.

Ky Fan (Oak Ridge, Tenn.).

Cristescu, Romulus. Sopra gli operatori biadditivi regolari. Rend. Mat. e Appl. (5) **15** (1956), 271–290 (1957).

This article continues the study of bilinear operators that the author has pursued in a number of earlier papers [see Acad. R. P. Romîne. Stud. Cerc. Mat. **6** (1955), 357–493; MR **18**, 404, and the literature cited in the review]. Terminology and notation are as in the review cited. The author here supposes that his biadditive operators are defined everywhere on $\mathfrak{X} \times \mathfrak{Y}$, and considers in particular regular operators, i.e., those that can be expressed as the difference of two non-negative biadditive operators. A number of simple properties of such operators, for various types of spaces $\mathfrak{Z}$, are obtained. *E. Hewitt.*

Holladay, John C. A note on the Stone-Weierstrass theorem for quaternions. Proc. Amer. Math. Soc. **8** (1957), 656–657.

In this note, it is proved that if A is a set of continuous quaternion-valued functions defined on a compact Hausdorff space X which satisfies the following conditions: (i) A is complete; (ii) given a quaternion q, the function $f(x)=q$ is in A; and (iii) if $f, g \in A$, then $fg, f+g \in A$; then A contains all continuous quaternion-valued functions if A distinguishes between every pair of distinct points of X. The proof depends on the following fact: $f \in A$ implies the real part of $f=(f-ifi-jfj-kfk)/4 \in A$. The well-known example of all analytic functions in the closed unit disc is mentioned to illustrate that in the theorem the word "quaternion" cannot be replaced by "complex".

W. A. J. Luxemburg (Toronto, Ont.).

Gel'fand, I. M. On some problems of functional analysis. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 6(72), 3–12. (Russian)

The author points out the widening domain of functional analysis and points to the applications of functional analysis to numerical analysis, theory of probability, theory of differential equations, and information theory. He believes that hydrodynamics (the flow of viscous fluids, compressible gases, theory of turbulence) and

theoretical physics (quantum theory and theory of elementary particles) will have a strong influence on the future course of development of functional analysis. Problems in the following fields are indicated: (1) linear topological spaces, (2) linear and quasi-linear partial differential equations, (3) measure theory, (4) rings of type P_1, and (5) hypergeometric functions. No specific references are given. *J. P. LaSalle* (Notre Dame, Ind.).

See also: Linear Algebra: Deprit. **Algebras:** Cartier. **Approximations, Orthogonal Functions:** Žuhovickiĭ and Stečkin. **Ordinary Differential Equations:** Delsarte and Lions; Brodskiĭ; Rehlickiĭ; Kato; Ladyženskaya. **Partial Differential Equations:** Séminaire Schwartz; Stummel; Gould. **Topological Groups:** Ellis; Reiter. **Banach Spaces, Banach Algebras, Hilbert Spaces:** Yosida; Fell; Umegaki; Birkhoff; Schäfke; Hartman; Leżański. **General Topology:** Pursell. **Programming, Resource Allocation, Games:** Duffin; Fan, Glicksberg and Hoffman.

Banach Spaces, Banach Algebras, Hilbert Spaces

★ **Yosida, Kôsaku. Isô-kaiseki. I. [Topological analysis. I.]** Iwanami Shoten, Tokyo, 1951. 2+339 pp. 680 yen.

This is a comprehensive book on functional analysis. In Chapters 1–5 fundamental concepts of functional analysis are introduced, and important concrete functional spaces are illustrated. Naturally these chapters are written in a modern way compared with Banach's book "Théorie des opérations linéaires". Chapter 6 deals with the theory of commutative normed rings of the Russian school, which will be a motivation for the following Chapters 7–10. Chapter 7 is concerned with Riesz's theorem on complete continuous operators in a Banach space. This is proved by the method of Cauchy's integration formula which was used by Gelfand to prove the representation theory of normed rings. In Chapter 8 spectral resolution of bounded normal operators is discussed on the basis of representation theory of normed rings. Typical examples of bounded hermitian and unitary operators are given and their spectral resolutions are illustrated. Chapter 9 deals with von Neumann's works on unbounded hermitian operators, accompanied by typical examples. Chapter 10 is concerned with closed operators on F-type spaces, and especially the canonical decomposition theorem of closed operators in a Hilbert space is proved. In Chapter 11 general notions on non-commutative normed rings are explained and their representations in a ring of linear operators on a Banach space are discussed. Again as applications of commutative normed rings Wiener's Tauberian theorem and other important theorems of Fourier analysis are proved. In Chapter 12 the author develops his theory of one-parameter semigroups of linear operators on a Banach space. From this point of view he shows P. Lévy's infinitely divisible laws, and the integrability of Fokker-Planck's equations. Chapter 13 deals with vector lattices. Many important theorems on functional analysis are viewed from this stand point. Finally a quick sketch of L. Schwartz's distribution theory is given in the appendix.

H. Anzai (Zbl. 45 (1955), 212).

Fell, J. M. G. Representations of weakly closed algebras. Math. Ann. 133 (1957), 118–126.

Let A be a C^*-algebra with identity element 1 and R a *-representation of A in a Hilbert space H. Call R "separable" if H is separable, and "faithful-irreducible" if there is no proper closed invariant subspace K of H such that the kernel of the restriction of R to K is equal to the kernel of R. If h is a central projection in A (i.e., $h^*=h$, $h^2=h$ and h belongs to the center of A), then hA is called a "direct summand" of A. A representation R is called a "direct summand representation" if, for any closed invariant subspace K of H, the restriction of R to K has for its kernel a direct summand of A. Now let A be an AW^*-algebra of Type I [for the definition of AW^*-algebra and associated notions cf. Kaplansky, Ann. of Math. (2) 53 (1951), 235–249; MR 13, 48]. The author shows that, for each central projection h of A, there exists (to within unitary equivalence) precisely one faithful-irreducible direct summand representation R of A whose kernel is $(1-h)A$. Denote this representation by $R(A, h)$. The main result of the paper can now be stated as follows: Let A be a purely infinite AW^*-algebra of Type I which satisfies the countable chain condition. Then any separable *-representation R of A is of the form $R \cong \sum_{h\in D} \varphi(h)\cdot R(A, h)$, where D is a family of pairwise orthogonal non-zero central projections in A whose sum is 1, φ is a one-to-one function on D whose values are cardinal numbers, and $\varphi(h)\cdot R(A, h)$ is the direct sum of $\varphi(h)$ copies of the representation $R(A, h)$. *C. E. Rickart.*

Umegaki, Hisaharu. Weak compactness in an operator space. Kōdai Math. Sem. Rep. 8 (1956), 145–151.

Several facts in integration theory are generalized in a straightforward manner to "non-commutative integration". Let (A, μ) be a gage space. A set S of continuous linear functionals on A is called equi μ-absolutely continuous if, given $\varepsilon>0$, there exists $\delta>0$ such that whenever p is a projection with $\mu(p)<\delta$, then $f(p)<\varepsilon$ for all $f \in S$. If K is a subset of $L_1(A, \mu)$, then equi μ-absolute continuity for K will mean equi μ-absolute continuity for the set S of functionals on A of the form $y\to\mu(xy)$, $x \in K$. Theorem (Vitali-Hahn-Saks): Let $\{f_n\}$ be a sequence of linear functionals on A, strongly continuous on the unit sphere of A. Suppose $\lim_{n\to\infty} f_n(p)$ exists for each projection p. Then $\{f_n\}$ is equi μ-absolutely continuous. Theorem (Lebesgue Completeness Theorem): Let $\mu(I)$ be finite. Then a subset K of $L_1(A, \mu)$ is weakly conditionally compact if and only if it is both equi μ-absolutely continuous and bounded in the $L_1(A, \mu)$ norm. Corollary: Let S be a bounded set of non-negative, completely additive linear functionals on A. Suppose $f(p_\alpha)\downarrow 0$ whenever the projections $p_\alpha \downarrow 0$. Then S is weak * conditionally compact. Corollary (Kakutani Completeness Theorem): If $0\leqq x_1<x_2 \in L_1(A, \mu)$, then $\{x | x_1\leqq x\leqq x_2\}$ is weakly compact. Theorem: Necessary and sufficient for a subset K of $L_1(A, \mu)^+$ to be weakly sequentially conditonally compact is that, for every sequence $p_1, p_2, \cdots$ of projections descending to 0, $\mu(xp_n)$ goes to 0 uniformly for x in K. Corollary: If A is a finite W^* algebra then $L_1(A, \mu)$ is weakly sequentially complete. *J. Feldman.*

Krabbe, G. L. Spectral isomorphisms for some rings of infinite matrices on a Banach space. Amer. J. Math. 78 (1956), 42–50.

Let l_p $(p>1)$ be the usual Banach space of complex sequences $c=\{c_n\}$ such that $\sum_{-\infty}^{\infty} |c_n|^p<\infty$, with the customary norm. Given a suitable function F on $[-\pi, \pi]$, let f be its sequence of Fourier coefficients and let $\langle F\rangle$ be the transformation on l_p defined by convolution with f; that is, $\langle F\rangle c=f*c$, where $(f*c)_n=\sum_{m=-\infty}^{\infty} f_{n-m}c_m$. If F is a

function of bounded variation on $[-\pi, \pi]$, then its total variation plus its uniform norm is a norm under which the space BV of all such functions becomes a Banach algebra. The author shows that the mapping $F \to \langle F \rangle$ is a bounded isomorphism of BV into the Banach algebra of all bounded linear transformations of l_p into itself. He also proves that if F is absolutely continuous then the spectrum of operator $\langle F \rangle$ is precisely the range of the function F, from which it follows that $\langle F \rangle^{-1}$ exists if and only if F^{-1} exists, and $\langle F \rangle^{-1} = \langle F^{-1} \rangle$. *L. H. Loomis.*

Harazov, D. F. On a class of operators which depend nonlinearly on a parameter. Dokl. Akad. Nauk SSSR (N.S.) **112** (1957), 819–822. (Russian)

The author considers, in a Hilbert space X, equations of the form

$$(*) \qquad x - A_0 x - \lambda\{A_1 x + \lambda \sum_{i=1}^{\infty} (\lambda - a_i)^{-1} H_i x\} = y,$$

where A_0 is self-adjoint and completely continuous, $I - A_0$ is positive definite, A_1 is self-adjoint and of finite Hilbert-Schmidt norm, each H_i is self-adjoint and of finite rank, the norms and traces of the H_i are uniformly bounded, $a_i H_i$ is negative semidefinite for each i and $\sum |a_i|^{-1} < +\infty$. The number λ is defined to be a characteristic value of the associated homogeneous equation if either (a) λ is a characteristic value in the ordinary sense, or (b) $\lambda = a_p$ for some p and there exist elements φ and ψ in X such that $\varphi \neq 0$, $H_p \varphi = 0$, and

$$\varphi - A_0 \varphi - a_p\{A_1 \varphi + a_p \sum_{i \neq p} (a_p - a_i)^{-1} H_i \varphi\} + a_p H_p \psi = 0.$$

The author obtains minimax properties of the characteristic values, expansion theorems for $A_1 f$, $H_i f$ and the solutions of (*), and some more complex expansion theorems arising from the introduction of an inner product space consisting of sequences $(y_0, y_1, y_2, \cdots)$ of elements of X such that

$$(y_0 - A y_0, y_0) - \sum_{i=1}^{\infty} a_i (H_i y_i, y_i) < +\infty.$$

F. Smithies (Cambridge, England).

Schieferdecker, Eberhard. Einige Sätze aus dem Ideenkreis des Schwarzschen Lemmas der Funktionentheorie in komplexen Banach-Räumen. Math. Ann. **132** (1957) 430–441.

Let $f(x)$ be defined in a domain $\mathfrak{G}$ of the complex B-space $\mathfrak{X}$ and have values in a similar space $\mathfrak{Y}$. Let $f(x)$ be F-holomorphic in $\mathfrak{G}$, that is, there exists to every $x \in \mathfrak{G}$ a linear bounded operator $f'(x)$ on $\mathfrak{X}$ to $\mathfrak{Y}$ such that for $h \in \mathfrak{X}$, $\|h\| \to 0$, one has $\|f(x+h) - f(x) - f'(x)h\| = o(\|h\|)$. Since the lemma of Schwarz holds for F-holomorphic functions the author asks what related theorems from classical function theory will also hold. He proves a number of estimates. Thus, if $\|f(x)\| \leq M$ for $\|x\| < R$ the inequality

$$\|f(x)\| \leq M \frac{M\|x\| + R\|f(\theta)\|}{MR + \|f(\theta)\| \, \|x\|}$$

holds; further, if $f(x)$ and its first $(n-1)$ derivatives vanish for $x = \theta$, then $\|f(x)\| \leq M[\|x\|/R]^n$. Both estimates are best possible. For the derivatives one has to be satisfied with slightly less than what the analogy with the case $\mathfrak{X} = \mathfrak{Y} = C^1$ would suggest. Thus, if $\mathfrak{X} = C^1$ but $\mathfrak{Y}$ arbitrary,

$$\|f'(\zeta)\| \leq \frac{R}{M(R^2 - |\zeta|^2)} \{M_2 - \sup |y^*[f(\zeta)]|^2\},$$

where $y^* \in \mathfrak{Y}^*$, $\|y^*\| = 1$, $|y^*[f'(\zeta)]| = \|f'(\zeta)\|$, $\|f(\zeta)\| \leq M$, $|\zeta| < R$, and this is the best possible. If $\mathfrak{X}$ is also arbitrary one gets, for instance,

$$\|f'(x)\| \leq \frac{2MR}{R^2 - \|x\|^2}.$$

The methods used extend to derivatives of higher order. *E. Hille* (New Haven, Conn.).

Kendall, David G.; and Moyal, J. E. On the continuity properties of vector-valued functions of bounded variation. Quart. J. Math. Oxford Ser. (2) **8** (1957), 54–57.

A function x from the real line to a weakly sequentially complete complex Banach space which is of bounded variation in the weak sense on each compact interval is strongly continuous except at a denumerable number of points and the limits $x(t \pm 0)$ exist in the strong topology at all points t. As a consequence every such function has a separable range and, if continuous on the right, is the sum of a continuous function and a step function. The proofs are elementary and use only some knowledge of unconditionally convergent series. *N. Dunford.*

Edwards, D. A. On the continuity properties of functions satisfying a condition of Sirvint's. Quart. J. Math. Oxford Ser. (2) **8** (1957), 58–67.

The results of the preceding review are proved under the assumption that for each finite interval (a, b) the set $E(a, b)$ of all sums of the form $\sum_{i=1}^{n} \{x(t_{2i}) - x(t_{2i-1})\}$ with $a \leq t_1 < t_2 < \cdots < t_{2n} \leq b$ is conditionally weakly compact. It is not assumed that the Banach space X in which $x(t)$ lies is weakly complete nor is it assumed that x is of bounded variation in the weak sense (although this latter condition is an immediate consequence of the stated assumption). That the new results form extensions of those proved by Kendall and Moyal is made evident by the theorem (proved in the paper) asserting that, in weakly complete spaces, the conditional compactness of $E(a, b)$ is equivalent to its boundedness. In order to modify the methods of Kendall and Moyal the author shows that for an arbitrary complex B-space X the series $\sum x_n$ is unconditionally convergent if and only if the set $\{\sum_{m=1}^{M} x_{n_m} : M \geq 1, 1 \leq n_1 < n_2 < \cdots < n_m < \infty\}$ is conditionally weakly compact. A number of other related results are demonstrated including a short proof of a theorem of Sirvint which gives the representation of the general weakly compact linear map from $C(a, b)$ to an arbitrary B-space. *N. Dunford* (New Haven, Conn.).

Jacobs, Konrad. Fastperiodizitätseigenschaften allgemeiner Halbgruppen in Banach-Räumen. Math. Z. **67** (1957), 83–92.

The author extends decomposition theorems of earlier papers [particularly that of Math. Z. **64** (1956), 298–338; MR **17**, 988]. The main theorem is: Let B be a Banach space and let G be a semigroup of linear transformations of B into B. Assume that B and its dual space B' can be so normed that each is strictly convex, the transformations of G and its dual G' are contractions, and at least one of B, B' is uniformly convex. Then B admits a decomposition into the direct sum of subsets P and F, where P is the set of vectors almost periodic under G and F is the set of vectors furtive under G. Paired with this there is derived an ergodic theorem which is a generalization of that of Alaoglu and Birkhoff. *G. A. Hedlund.*

Birkhoff, Garrett. Extensions of Jentzsch's theorem. Trans. Amer. Math. Soc. **85** (1957), 219–227.

The Jentzsch theorem referred to in the title asserts the existence of a positive eigenfunction with a positive eigenvalue for a homogeneous Fredholm integral equation with a continuous positive kernel [J. Reine Angew. Math. **141** (1912), 235–244] and may be considered as an analogue of a theorem of Perron [Math. Ann. **64** (1907), 1–76] concerning matrices with positive coefficients. The latter theorem was given a topological proof using Brouwer's fixed-point theorem by Alexandroff and Hopf [Topologie, Springer, Berlin, 1935, p. 480], a method which suggested giving a proof of the Jentzsch theorem based on Schauder's fixed-point theorem. This was done by M. Rutman [Mat. Sb. N.S. **8(50)** (1940), 77–96; MR **2**, 104] who at the same time generalized the theorem to linear completely continuous operators leaving a cone in Banach space invariant. For later results derived by operator methods concerning linear maps of a cone into itself, some of which do not assume complete continuity but only boundedness, we refer to a paper by Kreĭn and Rutman [Uspehi Mat. Nauk (N.S.) **3** (1948), no. 1(23), 3–95; MR **10**, 256; **12**, 341].

In the paper under review the problem of fixed directions for a bounded linear map P of a cone C into itself is attacked by introducing a projective metric under which the cone becomes a metric space (which, e.g., in the case that the intersection of C with a hyper-plane is an ellipsoid becomes the hyperbolic space). A natural norm $N(P)$ of P is defined and (under a certain completeness assumption concerning C) it is shown that if, for some integer r, $N(P^r)<1$, then for any direction x in C the iterations $P^r(x)$ converge geometrically to a unique fixed direction in C. The assumption $N(P^r)<1$ is, e.g., satisfied if the diameter of the image of C (in the projective metric) is finite.

Various applications of this general set-up are made; e.g., the existence of a unique positive eigenelement with positive eigenvalue for a "uniformly positive" bounded transformation P of a Banach lattice L into itself is proved; a decomposition of L connected with such a P is effected which gives some information about the spectrum of P. Finally a theorem concerning semigroups of "uniformly positive" bounded operators P_t $(t>0)$ of L into itself is derived. This theorem is connected with results on multiplicative processes by Everett and Ulam [Proc. Nat. Acad. Sci. U.S.A. **34** (1948), 403–405; MR **10**, 132].

E. H. Rothe (Ann Arbor, Mich.).

Schäfke, Friedrich Wilhelm. Zur Störungstheorie der Spektralzerlegung. Math. Ann. **133** (1957), 219–234.

The author discusses the perturbation of an isolated eigenvalue λ_0 in the spectrum of a self-adjoint operator F under the perturbing operator $G(\mu)=\sum_{n=1}^{\infty}\mu^n G_n$, where $\|G_n u\|\leq\gamma_n(\|u\|, \|Fu\|)$. The functions $\gamma_n(\xi, \eta)$ are defined, for ξ, $\eta>0$, as non-negative, non-decreasing and homogeneous functions. The estimates are improvements of previous ones obtained by Rellich [Math. Ann. **113**, 600–619 (1936), 677–685 (1937); **116** (1939), 555–570; **117** (1940), 356–382; **118** (1942), 462–484; MR **2**, 105; **6**, 71], Nagy [Comment. Math. Helv. **19** (1946), 347–366; MR **8**, 589], and Schröder [Math. Nachr. **10** (1953), 113–128; MR **16**, 264]. The method used rests on methods developed by the author in previous papers [Math. Nachr. **6** (1951), 109–124; Z. Angew. Math. Mech. **33** (1953), 255–259; MR **14**, 185; **15**, 474] and on successive approximations.

František Wolf (Berkeley, Calif.).

Hartman, Philip. Perturbation of spectra and Krein extensions. Rend. Circ. Mat. Palermo (2) **5** (1956), 341–354 (1957).

Methods of approximating spectral values of certain self-adjoint operators A on a Hilbert space H are given. The technique used is one of considering self-adjoint extensions of a sequence of restrictions of the original operator to a corresponding increasing sequence of closed linear manifolds of the original space, and of approximating spectral values of the original operator in terms of the spectral values of this sequence of extensions. This technique gives methods of determining when a spectral value is of infinite multiplicity and of approximating spectral values of the lower portions of the spectrums, i.e., the spectral values preceding the first cluster point of the spectrum or the first point of infinite multiplicity. This method generalizes one used by the author previously in a particular case [Duke Math. J. **15** (1948), 679–709; MR **10**, 376].

R. E. Fullerton (College Park, Md.).

Sz.-Nagy, Béla. Fortsetzungen linearer Transformationen des Hilbertschen Raumes mit Austritt aus dem Raum. Schr. Forschungsinst. Math. **1** (1957), 289–302.

Let H be a Hilbert space and T a bounded linear operator over H. Let H' be a Hilbert space containing H. In recent years a number of theorems have been proved which under certain conditions on T express T as a restriction to H of an operator of the form PS, where S is an operator of some restricted type (self-adjoint, unitary, normal, etc.) on H', and P is the projection of H' onto H. This allows one to investigate spectral representations of operators T of more general type than those usually considered. The author discusses a number of known theorems of this type and shows that they can all be deduced as special cases of the following general theorem. Let Γ be a * semigroup with unit ε (a semigroup with an involution * such that $\gamma^{**}=\gamma$, $\gamma_1{}^*\gamma_2{}^*=(\gamma_2\gamma_1)^*$) and let $\{T_\xi\}_{\xi\in\Gamma}$ be a family of bounded linear operators satisfying (a) $T_\varepsilon=I$; (b) $T_{\xi^*}=T_\xi{}^*$; (c) $\sum_\xi\sum_\eta (T_{\xi^*\eta}g_\eta, g_\xi)\geq 0$ for every family of elements $\{g_\xi\}_{\xi\in\Gamma}$, where $g_\xi=0$ for all but a countable family of $\xi\in\Gamma$; (d) for every family $\{g_\xi\}$ of the type in (c) and for every $\alpha\in\Gamma$,

$$\sum_\xi\sum_\eta (T_{\xi^*\alpha^*\alpha\eta}g_\eta, g_\xi)\leq C_\alpha{}^2\sum_\xi\sum_\eta (T_{\xi^*\eta}g_\eta, g_\xi)$$

for some $C_\alpha>0$. Then there exists a Hilbert space $H'\geq H$ and a family of bounded linear operators $\{D_\xi\}_{\xi\in\Gamma}$ on H' satisfying $D_\varepsilon=I$, $D_{\xi\eta}=D_\xi D_\eta$, $D_{\xi^*}=D_\xi{}^*$, and such that $T_\xi=PD_\xi$ for all $\xi\in\Gamma$. The space H' can be taken as minimal and in this case we have also (1) $\|D_\alpha\|\leq C_\alpha$, (2) if $T_{\xi\alpha\eta}=T_{\xi\beta\eta}+T_{\xi\gamma\eta}$ for fixed α, β, γ and all $\xi, \eta\in\Gamma$, then $D_\alpha=D_\beta+D_\gamma$, (3) if $T_{\xi\alpha_n\eta}\to T_{\xi\alpha\eta}$ weakly for all $\xi, \eta\in\Gamma$ and $\limsup_n C_{\alpha_n}<\infty$ then $D_{\alpha_n}\to D_\alpha$ weakly.

R. E. Fullerton (College Park, Md.).

Sz.-Nagy, B.; et Korányi, A. Relations d'un problème de Nevanlinna et Pick avec la théorie des opérateurs de l'espace hilbertien. Acta Math. Acad. Sci. Hungar. **7** (1956), 295–303. (Russian summary)

The authors give a new proof of the classical Nevanlinna-Pick theorem based on methods of functional analysis. The methods yield the following interesting analogue in the theory of linear transformations. For each s belonging to some set S in $|s|<1$, let $F(s)$ be a bounded linear transformation in a Hilbert space $\mathfrak{H}$. Let the kernel $K(s, t)=(1-s\bar{t})[F(s)+F^*(t)]$ be of positive type in the sense that $\sum_i\sum_j (K(s_i, s_j)u_i, u_j)\geq 0$ for every finite system of points $s_i\in S$ and elements $u_i\in\mathfrak{H}$; and let $F(0)=I$,

$0 \in S$. Then $F(s)$ may be extended to $|s|<1$ by $F(s)=\mathrm{pr}\,\frac{1}{2}(I+sU)/(I-sU)$, where U is a unitary transformation in an extended Hilbert space $\hat{\mathfrak{H}}$ of $\mathfrak{H}$, and where pr is the operator which projects $\hat{\mathfrak{H}}$ onto $\mathfrak{H}$. *A. J. Lohwater.*

Altman, M. A fixed point theorem in Banach space. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 89–92, IX. (Russian summary)

Let X be a Banach space and Q the sphere with center at θ and radius r. If F is a completely continuous operator defined on Q and if $\|x-F(x)\|^2 \geq \|F(x)\|^2-\|x\|^2$ for each x on the boundary of Q, then F has a fixed point. Applications to non-linear integral equations are sketched.

E. G. Begle (New Haven, Conn.).

Altman, M. A fixed point theorem in Hilbert space. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 19–22, III. (Russian summary)

The following theorem is announced: Let H be a real separable Hilbert space and, for a fixed $r>0$, let Q be the set of all $x \in H$ with $\|x\| \leq r$. Let F be a weakly closed map of Q into a bounded subset of H such that for each x for which $\|x\|=r$, $(F(x), x) \leq (x, x)$. Then F has a fixed point.

Applications to non-linear integral equations are also stated. *E. G. Begle* (New Haven, Conn.).

Altman, M. On linear functional equations in Hilbert space. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 223–227, XIX. (Russian summary)

Let H be a real separable Hilbert space with a complete orthonormal system $\{e_i\}$. This paper is concerned with the equation (1) $x-Fx=y$, where F is a weakly continuous (generally non-linear) operator defined on $Q=\{x \in H : \|x\| \leq r\}$ with values in H. If there exists a positive number $M<1$ such that $\|x\|=r$ implies $(Fx, x) \leq M(x, x)$, then for every $y \in H$ with $\|y\| \leq (1-M)r$: (a) equation (1) has at least one solution $x \in Q$; (b) for each n, there exists an $x_n \in Q$ satisfying $x_n=\sum_{i=1}^{n}(Fx_n+y, e_i)e_i$; (c) every weak cluster point of $\{x_n\}$ is a solution of (1). The author also discusses the special case when F is a bounded linear operator defined on H, together with applications to linear integral equations and infinite systems of linear equations. The paper is closely related to the one reviewed above.

Ky Fan (Oak Ridge, Tenn.).

Altman, M. On the method of orthogonal projection. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 229–231, XIX. (1 plate). (Russian summary)

The paper gives a computational scheme for finding an approximate orthogonal projection in a linear subspace of an element of a Hilbert space. This is closely related to the author's two other recent papers [same Bull. **5** (1957), 23–27, 29–33; MR **19**, 41]. *Ky Fan.*

Leżański, T. Approximate method of calculating characteristic elements and values. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 1–3, I. (Russian summary)

A method is described for approximating characteristic values and elements of a symmetric operator A on a complex Hilbert space. The method described has the advantage of using calculations which do not depend upon the calculated values of other characteristic values and hence are not affected by inaccuracies in previous approximations. *R. E. Fullerton* (College Park, Md.).

Leżański, T. Approximate solution of a linear equation. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 5–6, I–II. (Russian summary)

Let $\mathscr{H}$, $\mathscr{H}_1$ be Hilbert spaces, A a bounded linear operator from $\mathscr{H}$ to $\mathscr{H}_1$, $\varphi_0 \in \mathscr{H}_1$, $Ax=\varphi_0$ uniquely solvable. The author constructs a function $x(t)$ from the positive reals to $\mathscr{H}$ such that $\lim_{t\to\infty}\|Ax(t)-\varphi_0\|=0$, and is thus able to approximate the root of the given equation with values of $x(t)$ for large t.

R. E. Fullerton.

Leżański, T. Approximate calculation of the minimum of a convex functional. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 7–9, II. (Russian summary)

Let $\Phi(x)$ be a function on a Hilbert space $\mathscr{H}$, real valued, continuous, and bounded from below. Define grad $\Phi(x)$ by $\Phi(x+h)-\Phi(x)=(\text{grad }\Phi(x), h)+R(x, h)$, where $R(x, h)=O(\|h\|)$. Assume for each $x_0 \in \mathscr{H}$ the existence of a differentiable function $x(t):[0, \infty)\to\mathscr{H}$ such that $x'(t)=-\text{ grad }\Phi(x(t))$ for all t, and with $x(0)=x_0$. The author states a theorem giving a bound in terms of t on $\Phi(x(t))-\inf_{x\in\mathscr{H}}\Phi(x)$ that approaches 0 as $t\to\infty$. Thus, for a sufficiently large value of t, $x(t)$ approximates a minimizing point for Φ. *R. E. Fullerton.*

See also: Sequences, Series: Green; Roberts. **Approximations, Orthogonal Functions:** Zuhovickiĭ and Stečkin; Vertgeĭm; Hummel. **Ordinary Differential Equations:** Brodskiĭ; Rehlickiĭ; Orlov; Kato; Ladyženskaya. **Partial Differential Equations:** McKean; Gould. **Topological Groups:** Reiter. **Numerical Methods:** Samokiš.

TOPOLOGY

General Topology

Banaschewski, Bernhard. Zur Existenz von universellen Überlagerungen. Math. Nachr. **15** (1956), 175–180.

A generalization is given of the well-known theorem on the existence of simply connected covering spaces [cf. C. Chevalley, Theory of Lie groups, I, Princeton, 1946; MR 7, 412]; roughly speaking, if E is locally connected, $\mathfrak{B}$ a covering of E consisting of connected open sets, then there exists a (unique) "largest" covering space which evenly covers every $V \in \mathfrak{B}$. The proof is quite different from that in Chevalley's book.

M. Katětov.

Pursell, Lyle E. The ring $C(X, R)$ considered as a subring of the ring of all real-valued functions. Proc. Amer. Math. Soc. **8** (1957), 820–821.

Denote by R^X the ring of all functions from X to the reals, and by $C(X)$ its subring of continuous functions. An ideal in $C(X)$ is said to be fixed if its members have a common zero. It is known that an automorphism of $C(X)$ need not send fixed ideals into fixed ideals. The author shows, however, that the class of fixed ideals is invariant under automorphisms of $C(X)$ that have extensions to automorphisms of R^X. Consequently, for any completely regular spaces X and Y, the existence of an isomorphism of R^X onto R^Y that maps $C(X)$ onto $C(Y)$ implies that X and Y are homeomorphic. *M. Jerison.*

Isiwata, Takesi. Some characterizations of countably compact spaces. Sci. Rep. Tokyo Kyoiku Daigaku Sect. A. **5** (1956), 185–189.

It is proved that (1) a normal space X is countably compact if and only if (*) for any countable directed set $\{f_n\}$ of continuous (real-valued) functions, pointwise convergence to a continuous f implies quasi-uniform convergence, (2) a completely regular X is countably compact if and only if every continuous transformation of X into a space satisfying the first axiom of countability is closed. {Reviewer's remark: (1) follows at once from (1'): X is pseudocompact if and only if (*) holds; in (1'), "if" is clear (for an unbounded g, $g/n \to 0$ is not quasi-uniform), "only if" follows easily, e.g., from exercise 4 in § 2 of N. Bourbaki, Eléments de mathématique, Topologie générale, Chap. X, [Actualités Sci. Ind., no. 1084, Hermann, Paris, 1949; MR **12**, 40].} *M. Katětov.*

Šik, František. Die Struktur der Menge der Topologien mit vorgeschriebenen Gestirnen der Punkte. Publ. Fac. Sci. Univ. Masaryk **1955**, 459–472. (Czech. Russian and German summaries)

Unter der Topologie auf der Menge P ($\neq\emptyset$) versteht man eine beliebige Abbildung u der Menge 2^P in sich. In der vorliegenden Arbeit wird vor allem festgesetzt, dass die Menge $\mathfrak{M}$ aller Topologien, für die u monoton ist, einen vollständig distributiven Verband bildet (mit dem kleinsten Elemente v_0 und mit dem grössten w_0) und dass die Menge $\mathfrak{N}$ aller Topologien u, für die $v_0 \leq u \leq w_0$ gilt, eine vollständig distributive (und also atomische) Boolesche Algebra bildet, für die $\mathfrak{M} \subset \mathfrak{N}$ gilt.

Für die Untersuchung der Struktur des Verbandes $\mathfrak{M}$ sind auf dem Verbande $\mathfrak{M}$ zwei transfinite Folgen der Verbände konstruiert, in deren Kardinalprodukt man den Verband $\mathfrak{M}$ isomorph abbilden kann. Im Falle der unendlichen Menge $\mathfrak{M}$ sind die Mächtigkeiten der Faktoren des erwähnten Kardinalproduktes und daher die Mächtigkeiten der Mengen $\mathfrak{M}$ und $\mathfrak{N}$ bestimmt.

Aus der Zusammenfassung des Autors.

Sinclair, Annette. Pseudo locally compact spaces. Proc. Amer. Math. Soc. **8** (1957), 215–222.

The author introduces the following definition: A topological space $(R, \mathscr{R})$ will be said to be $\mathscr{R}'$-pseudo locally compact if any neighborhood by the $\mathscr{R}$-topology of an arbitrary point p of R contains some neighborhood $N_p(R, \mathscr{R})$ by the $\mathscr{R}$-topology such that the point set $\overline{N}_p(R, \mathscr{R})$ is compact by the $\mathscr{R}'$-topology. In order that pseudo local compactness be a somewhat weaker property than ordinary local compactness it is desirable that the topology $\mathscr{R}'$ be weaker than $\mathscr{R}$. Otherwise it is possible that $(R, \mathscr{R})$ be locally compact but not $\mathscr{R}'$-pseudo locally compact. Some general results about pseudo local compactness are proved, and are then applied to spaces of functions meromorphic in a region R. This set of functions is topologized in two different ways, and it is shown that the space with respect to the first topology (which seems to be of little function-theoretic interest) is pseudo locally compact with respect to the second topology (which is of more interest). {It was not clear to the reviewer whether $\mathscr{R}'$-pseudo local compactness is to serve as a property of the space $(R, \mathscr{R})$, as the wording would seem to indicate, or of the space $(R, \mathscr{R}')$. Three final remarks: compactness as used in the article is sequential compactness; metric spaces are introduced in which the metric function is not defined for certain pairs of points; the letter R, as both a Roman capital and a script capital, is used with a number of quite different meanings throughout the article.} *W. J. Thron* (Boulder, Colo.).

Ciarrapico, Lucia. Sugli spazi topologici generali soddisfacenti ad un assioma di non-crescenza ristretto. Rend. Mat. e Appl. (5) **13** (1955), 282–293.

La classe des espaces $(\mathscr{V})$ (au sens de M. Fréchet) dont le conjugué (au sens de L. Geymonat) est encore un espace $(\mathscr{V})$, coïncide avec la classe des espaces $(\mathscr{V})$ vérifiant l'axiome suivant, dit de non-croissance restreinte. (B) Pour tous sous-ensembles $A \neq \emptyset$ et B de l'espace on a: $\overline{A \cup B} - (A \cup B) \subset \overline{A} - A$; autrement dit, pour tous sous-ensembles $E \neq \emptyset$ et F de l'espace on a:

$$[E \subset F] \Rightarrow [\overline{F} - F \subset \overline{E} - E].$$

Cette classe est identique à la classe de tous les espaces susceptibles d'être définis comme suit: On se donne un ensemble S quelconque (l'ensemble des points de l'espace), et un ensemble quelconque $H \subset S$; on pose, pour tout ensemble non vide $E \subset S$: $E = E \cup H$, et de plus: $\bar{\emptyset} = \emptyset$. Le conjugué d'un espace de cette classe est un espace de la même classe, qui est obtenu en conservant S et changeant H en $S - H$. Les espaces de la classe envisagée satisfont aux quatre axioms de Kuratowski. L'axiome (B) est même la condition nécessaire et suffisante à laquelle doit satisfaire un espace topologique (au sens de M. Fréchet) pour que le conjugué de cet espace soit un espace $(\mathscr{V})$. *A. Appert.*

Behrend, F. A. Uniformizability and compactifiability of topological spaces. Math. Z. **67** (1957), 203–210.

Criteria are given for the uniformizability of a topological space S. A string Σ is defined as a family of subsets of S satisfying: (i) Every element of Σ is open or closed; (ii) Σ is fully ordered with respect to the inclusion relation, it has a least element A, which is closed, and a greatest element B, which is open; (iii) if $A_1 \in \Sigma$ is closed, $B_1 \in \Sigma$ is open, and $A_1 \subset B_1$, then there exists a closed $A_2 \in \Sigma$ and an open $B_2 \in \Sigma$ such that $A_1 \subset B_2 \subset A_2 \subset B_1$. The principal result of the paper is the proof that the following conditions (I)–(IV) are equivalent. (I) Given any open neighborhood $N(x)$ of any point $x \in S$ there exists a string of neighborhoods of x with greatest element $N(x)$. (II) There exists a relation $<$ between subsets of S with the following properties: (II.1) If $A < B$, then A is contained in the interior of B, (II.2) if $A < B$ then there exist A_1, B_1 such that $A < B_1 \subset A_1 < B$, (II.3) given any open neighborhood $N(x)$ of any point $x \in S$, there exists a neighborhood $M(x)$ such that $M(x) < N(x)$, (II.4) if $A < B$, then the complement of B precedes the complement of A. (III) S is uniformizable. (IV) There exists in S a family Δ of filters with the following properties: (IV.1) The neighborhood filter ϕ_x of any $x \in S$ belongs to Δ, (IV.2) if $\alpha \in \Delta$, $B \in \alpha$, then there exists $A \in \alpha$ such that every $\beta \in \Delta$ which is compatible with A contains B, (IV.3) every open filter of S is compatible with a filter $\alpha \in \Delta$. The author points out that (I) replaces Bourbaki's axiom (O_{IV}) which characterizes uniformizability in terms of continuous real functions on S. [See § 1, no. 5, of N. Bourbaki, Elements de Mathématique, Topologie générale, Chap. IX, Actualités Sci. Ind., no. 1045, Hermann, Paris, 1948; MR **10**, 260.]

In the proofs of these results no appeal is made to mappings of S into other spaces. In particular, no use is made of the real numbers. Inasmuch as a topological space is compactifiable if and only if it is a uniformizable

T_0-space, each criterion obtained, together with the condition that S be a T_0-space, gives a criterion for compactifiability.

Ky Fan and N. Gottesman have proven [Nederl. Akad. Wetensch. Proc. Ser. A. **55** (1952), 504–510; MR **14**, 669] that a topoligical space S is compactifiable if S is regular and possesses a normal base, i.e., a base Σ of open sets satisfying (1) if $A, B \in \Sigma$, then $A \cap B \in \Sigma$, (2) if $A \in \Sigma$, then the complement of $\bar{A}$ is an element of Σ, (3) for any open subset U of S and any $A \in \Sigma$ such that $\bar{A} \subset U$, there exists $B \in \Sigma$ such that $\bar{A} \subset B \subset \bar{B} \subset U$. The present author shows, using (II), that conditions (1) and (2) are not required for this conclusion. *D. W. Hall.*

Morita, Kiiti; and Hanai, Sitiro. Closed mappings and metric spaces. Proc. Japan Acad. **32** (1956), 10–14.

Suppose throughout that f is a closed continuous mapping of a T_1 space X onto a topological space Y. The principal theorems are: (Th. 1) If X is metrizable, Y is metrizable if and only if each inverse set $f^{-1}(y)$ $(y \in Y)$ has a compact frontier in X, or (Th. 4) if and only if Y is first countable. This includes recent theorems of Hanai [same Proc. **30** (1954), 285–288; MR **16**, 275] and Balanchandran [Duke Math. J. **22** (1955), 461–464; MR **17**, 392]. (Th. 3) If X is normal, or collectionwise normal, so is Y; if each $\mathrm{Fr}(f^{-1}(y))$ is compact, and X is paracompact, so is Y. E. Michael has shown that the hypothesis that $\mathrm{Fr}(f^{-1}(y))$ be compact is superfluous here [see the paper reviewed second below]. The case of normality in Th. 3 is well known. Theorems 1 and 4 have been announced independently by the reviewer [Bull. Amer. Math. Soc. **61** (1955), 309]. *A. H. Stone* (Manchester).

Stone, A. H. Metrizability of decomposition spaces. Proc. Amer. Math. Soc. **7** (1956), 690–700.

Let f be a continuous map from a metric space S onto a topological space E, with f quasi-compact (i.e. $U \subset E$ is open whenever $f^{-1}(U)$ is open); this paper studies conditions for E to be metrizable. The first section completely settles the case (previously considered by I. A. Vainstein, G. T. Whyburn, and S. Hanai) where f is closed, by proving that then each of the following two conditions is equivalent to the metrizability of E: (1) E satisfies the first countability axiom; (2) for each p in E, $f^{-1}(p)$ has a compact frontier. [These conditions were obtained independently, with earlier publication but later submission, in the paper reviewed above.] The later sections obtain sufficient conditions for some cases where f is not assumed closed; although these conditions are not necessary, examples show that none of them can be simply omitted. Two of these later theorems generalize results of A. V. Martin [Duke Math. J. **21** (1954), 463–469; MR **15**, 977]. *E. Michael* (Princeton, N.J.).

Michael, E. Another note on paracompact spaces. Proc. Amer. Math. Soc. **8** (1957), 822–828.

L'auteur améliore les critères de paracompacité donnés par lui dans un travail antérieur [mêmes Proc. **4** (1953), 831–838; MR **15**, 144]: il montre qu'un espace régulier E est paracompact si tout recouvrement ouvert de E admet un recouvrement fermé plus fin $\mathscr{R}$ tel que toute réunion d'ensembles de $\mathscr{R}$ soit encore un ensemble fermé. A l'aide de ce critère il prouve que tout espace quotient d'un espace paracompact par une relation d'équivalence fermée est paracompact. *J. Dieudonné* (Evanston, Ill.).

Terehova, N. P. Completeness of the space of subsets. Voronež. Gos. Univ. Trudy Sem. Funkcional. Anal. no. 1 (1956), 73–75. (Russian)

Let R be a metric space with metric ρ, and let $\hat{R}$ be the class of bounded subsets of R. It is shown that R is complete with respect to ρ if and only if $\hat{R}$ is complete with respect to the metric θ defined by $\theta(M_1, M_2) = \max\{\sup_{x \in M_1} \rho(x, M_2), \sup_{y \in M_2} \rho(y, M_1)\}$, with $M_1, M_2 \in R$ and $\rho(x, M) = \inf_{y \in M} \rho(x, y)$. *V. E. Beneš.*

Słowikowski, W. Elementary sequences in almost-metric spaces. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 109–112, X–XI. (Russian summary)

An almost metric space is a set with sufficiently many pseudometrics. Let X be such a space with the family Ξ of pseudometrics ρ_ξ, $\xi \in \Xi$, and topology defined as usual from the pseudometrics. Let N be the natural numbers and $\Lambda = \Xi \times N$. The set Ξ is supposed directed by the natural order, so Λ can be directed. A function on Λ to X is called an e-sequence. The e-sequence x is called a Cauchy sequence if for every $\lambda = (\xi, n) \in \Lambda$ there is a $\mu(\lambda) \in \Lambda$ scuh that $\rho_\xi(x(\lambda'), x(\lambda'')) < 1/n$ if $\lambda', \lambda'' \geqq \mu(\lambda)$. A subsequence is a composite function $x \circ \nu$: $\Lambda \to \Lambda$ such that $\nu(\lambda) \geqq \lambda$. Theorem: X is compact if and only if every e-sequence has a convergent subsequence; X is totally bounded if and only if every e-sequence has a Cauchy subsequence. *M.E. Shanks.*

Mrówka, S. On almost-metric spaces. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 123–127, XII. (Russian summary)

This paper states theorems whose proofs are to appear in Fund. Math. The author is interested in extending to almost metric spaces (see the preceding review for definition and notation) familiar theorems concerning sequences in metric spaces. His principal tool is called an m-sequence. If the set Ξ of pseudometrics has power m, let N^m be all functions α defined on Ξ with values in N such that $\alpha(\xi) \neq 0$ for only finitely many ξ. An m-sequence is a function mapping N^m into X. Unfortunately, many of the theorems are false as stated, as may be seen by taking $m = 1$. They may be true if for m one uses $\max(\aleph_0, |\Xi|)$. *M. E. Shanks* (Princeton, N.J.).

Mrówka, S. Remark on locally finite systems. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 129–132, XII. (Russian summary)

With the notation of the preceding reviews the author proves: A normal space is m-almost-metrizable if and only if it has an m-basis, where by the latter is meant a basis which is the union of at most m locally finite systems. The proof depends on the results of the paper reviewed above. *M. E. Shanks* (Princeton, N.J.).

Goetz, A. Invariante Metriken in homogenen Räumen. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 139–140, XII–XIII. (Russian summary)

We are given a topological space M and a group G of homeomorphisms of M which is transitive, which is such that the function xP, $x \in G$, $P \in M$, is continuous in both variables, and such that the mapping xP, P fixed, of G on M is open. A neighborhood U of a point P is said to be sphere-like if $xU = U$ for each x in G for which $xP = P$. The following theorem is stated: There is a metric on M, invariant with respect to G, if and only if 1) each point of M has a complete family of sphere-like neighborhoods, 2) M satisfies the first countability axiom. *E. G. Begle.*

Egorov, Vl. On metric dimension of point sets. Dokl. Akad. Nauk SSSR (N.S.) **112** (1957), 804–805. (Russian)

The metric dimension, dm A, of an arbitrary subset A of Euclidean space E^n is defined as the smallest r such that A has arbitrarily small displacements into a locally finite polyhedron of dimension r. Sitnikov gave an example with dm $A=1$, dim $A=2$ [Mat. Sb. N.S. **37(79)** (1955), 385–434; MR **17**, 1120]. For a metric space R one can define dm R by the usual covering definitions, using the family of ε-coverings, for all ε, where an ε-covering is one all of whose sets have diameter $<\varepsilon$. Metric dimension is invariant under uniform (in both directions) homeomorphisms. The author announces several characterizations of dm, all involving uniformity, in terms of: (a) coverings, locally finite in E^n (each $x \in E^n$ has a neighborhood that meets only a finite number of sets; (b) Lebesgue coverings (the covering has a refinement consisting of the ε-neighborhoods of some other covering, for some $\varepsilon>0$); (c) uniformly continuous essential maps into a simplex; (d) separation of sets; (e) extendability of uniform cocycles. *H. Samelson* (Ann Arbor, Mich.).

Nagata, Jun-iti. Note on dimension theory. Proc. Japan Acad. **32** (1956), 568–573.

Necessary and sufficient conditions, expressed in terms of sequences of open coverings, are given for the metrizability and n-dimensionality of a space [for similar results, cf. J. Nagata, same Proc. **32** (1956), 166–170, 237–240; MR **18**, 224; **19**, 156]. It is proved that, for a metrizable R, dim $R\leq n$ implies that R is homeomorphic with a subset of the product of $n+1$ metrizable R_i, dim $R_i\leq 1$; dim $R\leq n$ if and only if there is a metric of R constructed (in a specified way) from $n+1$ "non-Archimedean parametrics", i.e. symmetric non-negative functions ρ_i on $R\times R$ such that $\{y|\rho_i(x, y)<\varepsilon\}$ are open, $\rho_i(x, y)\leq\max[\rho_i(x, z), \rho_i(y, z)]$. *M. Katětov.*

Smirnov, Yu. M. Geometry of infinite uniform complexes and δ-dimensionality of point sets. Mat. Sb. N.S. **40(82)** (1956), 137–156. (Russian)

In a previous note [Dokl. Akad. Nauk SSSR (N.S.) **95** (1954), 717–720; MR **16**, 845], the author stated (without proof) that, for a subset A of the Euclidean n-space E^n, $\delta dA=n$ (where δdA denotes the δ-dimension of the proximity space A) if and only if there exists $r>0$ such that, for any $\varepsilon>0$, the set $\{y|\rho(y, A)<\varepsilon\}$ contains an r-sphere. The present article contains a proof, based on properties of uniform complexes (a notion introduced by the author) in E^n; an Euclidean simplicial (closed simplexes are considered) complex K in E^n is called a Lebesgue complex if, for some $\alpha>0$, the distance of any two non-intersecting simplexes of K is $\geq\alpha$; a Lebesgue complex K is called uniform if the set of the diameters of the simplexes of K is bounded. *M. Katětov.*

Lelek, A. Sur les décompositions en trois continus. Bull. Acad. Polon. Sci. Cl. III. **4** (1956), 511–513.

La but de cette note est de préciser la relations entre les deux propriétés suivantes: (S) X étant décomposé en trois continus, $X=K_1+K_2+K_3$, tels que $K_m K_n\neq 0$ pour $m, n=1, 2, 3$, on a $K_1K_2K_3\neq 0$, et (U) X étant decomposé en deux continus, $X=C_1+C_2$, leur partie commune C_1C_2 est un continu. D'après les résultats de cette note, les propriétés (S) et (U) sont équivalentes pour tous les X localement connexes, et (U) éntraîne (S) pour tous les autres X. Mais (S) n'entraîne pas (U) en général. *M. Kondô* (Tokyo).

Michael, J. H. Continuous mapping of subsets of the Euclidean n-sphere. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 133–137, XII. (Russian summary)

Soient A et B deux sous-ensembles fermés d'un n-sphère euclidean S^n tel que $A\neq S^n$, et soit f une application continue telle que $f(A)\supseteq B$, $f(\mathrm{Fr}(A))\cap f(\mathrm{Int}(A))=0$ et qui est homéomorphique sur $\mathrm{Fr}(A)$. Dès lors, en se servant de quelques résultats sur le degré des applications continues, l'auteur démontre qu'on a $f(\mathrm{Fr}(A))=\mathrm{Fr}(B)$ et donc $f(\mathrm{Int}(A))=\mathrm{Int}(B)$. C'est une réponse affirmative pour une conjecture de M. K. Kuratowski. *M. Kondô.*

Bing, R. H. Approximating surfaces with polyhedral ones. Ann. of Math. (2) **65** (1957), 456–483.

The main result proved is the following theorem. Let S be a triangulated 3-manifold. Let M be a 2-manifold with boundary, lying in S, and let f be a non-negative continuous function defined on M. Then there is a homeomorphism h of M onto $M'\subset S$ such that (i) if $f(x)>0$, then M' is locally polyhedral at $h(x)$ and (ii) the distance from x to $h(x)$ is never greater than $f(x)$. In particular, if $f=\varepsilon>0$, the theorem asserts that a 2-manifold with boundary can always be approximated arbitrarily closely by a polyhedral 2-manifold with boundary. For the special case in which M is semi-locally tamely imbedded, the above theorem is a corollary of a theorem of the reviewer [Ann. of Math. (2) **56** (1952), 96–114, Th. 2; MR **14**, 72]. But this is not true of the general case. Moreover, the argument is entirely self-contained; in fact, the necessary algebraic concepts are developed ab initio, for the special cases in which they are needed. *E. E. Moise.*

Harrold, O. G., Jr. Some consequences of the approximation theorem of Bing. Proc. Amer. Math. Soc. **8** (1957), 204–206.

The approximation theorem proved in the paper reviewed above is used to simplify the statements of "properties $\mathscr{P}$ and $\mathscr{Q}$" in Harrold, Griffith, and Posey [Proc. Nat. Acad. Sci. U.S.A. **40** (1954), 235–237; MR **15**, 816]. This is followed by some remarks about arcs and curves that do not lie on any 2-cell. *R. H. Fox.*

Whitehead, J. H. C. Note on the condition n-colc. Michigan Math. J. **4** (1957), 25–26.

Locally compact spaces are considered for which each point has a neighborhood basis which is well-ordered with respect to inclusion. It is shown that if X is such a space, and x is a point of X, then X is n-colc at x if and only if $H_n(X, X-x)=0$.

Here the group $H_n(X, X-x)$ is defined to be $\mathrm{Lim}\, H_n(X_\lambda, B_\lambda)$, (X_λ, B_λ) a compact pair contained in $\overrightarrow{(X, X-x)}$. *E. G. Begle* (New Haven, Conn.).

Wilder, R. L. Some consequences of a method of proof of J. H. C. Whitehead. Michigan Math. J. **4** (1957), 27–31.

The method of proof used by Whitehead in the note reviewed above is exploited to obtain a number of results of which the following is a typical example: Let A and B be closed subsets of a compact space S such that every r-cycle on A bounds on some compact subset of $S-B$. Then, if there is a basis for the open neighborhoods of B which is well-ordered by inclusion, there exists a neighborhood U of B such that every r-cycle on A bounds on $S-U$.

Next, by a slight refinement of Whitehead's argument, a number of results are obtained which relate two kinds of local Betti numbers. *E. G. Begle.*

Plunkett, Robert L. A fixed point theorem for continuous multi-valued transformations. Proc. Amer. Math. Soc. 7 (1956), 160–163.

The author employs the following characterization of continuity for multi-valued transformations. A multi-valued transformation F of a space X into a compact Hausdorff space Y is defined to be continuous provided for each $x_0 \in X$ it is true that: (1) $F(x_0)$ is closed, (2) V open and containing $F(x_0)$ implies that there exists an open set U' containing x_0 such that, if $x \in U'$, then $F(x) \subset V$, and (3) $y_0 \in F(x_0)$, $y_0 \in V$, and V open imply that there is an open set U'' containing x_0 such that, if $x \in U''$, then $F(x) \cdot V \neq \emptyset$. He proves: A nondegenerate continuous curve X has the fixed point property under all continuous multi-valued transformations of X into X if and only if it is a dendrite. *W. T. Puckett.*

Hamilton, O. H. Fixed points for certain noncontinuous transformations. Proc. Amer. Math. Soc. 8 (1957), 750–756.

A mapping T of a space A into a space B is called a connectivity map if the mapping: $g(P)=(p \times T(p))$ of A into $A \times B$ carries connected sets in A onto connected sets in $A \times B$. It is shown in this paper that a connectivity map of an n-cube into itself always has a fixed point. The proof is in two steps, the first of which shows that a connectivity map T of an n-cell, $n \geqq 2$, into itself is peripherally continuous, in the sense that for each point p, each neighborhood U of $T(p)$, and each neighborhood V of p there is a neighborhood D of p, $D \subset V$, such that T carries the boundary of D into U. Next, it is shown that any peripherally continuous map of an n-cell, $n \geqq 2$, into itself has a fixed point. From these two theorems the result follows if $n \geqq 2$, and the result is immediate if $n=1$. The paper concludes with some interesting examples. *E. G. Begle* (New Haven, Conn.).

Bourgin, D. G. Some mapping theorems. Rend. Mat. e Appl. (5) **15** (1956), 177–189.

In this note the author proves theorems proved originally by C. T. Yang, E. E. Floyd, G. R. Livesay and J. W. Jawokowski, or generalizations thereof. Among these are the following. Theorem 1: If M separates the origin from infinity in 3-dimensional euclidean space, there are three points on M at the same distance from the origin which form a figure similar to a given figure on the unit sphere. Theorem 2: If X is acyclic and compact with a homeomorphism p of prime power order, then p cannot be fixed-point free. Theorem 3: Let $f(r)$ be a map from $O(3)$ into the real line; then given two rotations R_1 and R_2 of order 4, with $f(r)=f(rR_i^2)$ $(i=1, 2)$, there exists an element r in $O(3)$ satisfying $f(r)=f(rR_1)=f(rR_2)$. In addition to these, there are several interesting lemmas, but frequent references to other papers and typographical errors make the paper hard to read. *H. Yamabe* (Minneapolis, Minn.).

See also: Lie Groups and Algebras: Mostow.

Algebraic Topology

★**Dowker, C. H. Lectures on sheaf theory. Notes by S. V. Adavi and N. Ramabhadran.** Tata Institute of Fundamental Research, Bombay, 1956. v+212+iv+iii pp. (mimeographed)

This course of lectures follows very closely the exposition of the theory of sheaves (faisceaux) given by H. Cartan in the Séminaire de topologie algébrique de l'Ecole Normale Supérieure, 1950/51 [MR **14**, 670]. The cohomology groups $H^q(X, \mathcal{S})$ of a space X (paracompact and normal) with coefficients in a sheaf $\mathcal{S}$ are defined, and the uniqueness theorem is proved. There is a partial treatment of Poincaré duality, and the concluding chapters deal briefly with coherent sheaves. Fibre spaces and spectral sequences are not treated, but there is a section on double complexes which is applied to prove Leray's theorem on acyclic coverings. Unlike the Cartan seminars, however, the exposition does not aim at conciseness. Many parts of the theory are treated in much greater detail, most calculations are given in full, and much standard algebra is developed ab initio. In this respect the work may serve a very useful purpose.

Perhaps the most characteristic feature of the lectures is the way each new notion is analysed very carefully, its peculiarities examined on "bad spaces", and counter-examples given whenever possible. For example, both fine and locally fine sheaves are defined, and it is shown that they coincide on paracompact normal spaces but not otherwise. There is also a very careful treatment of presheaves as opposed to sheaves. The lectures on coherent sheaves are dealt with in the same analytical spirit, and there is no attempt to go far into the applications to algebraic geometry or complex manifolds. *M. F. Atiyah* (Cambridge, England).

★**Koszul, J. L. Faisceaux et cohomologie.** Instituto de Matemática Pura e Aplicada do C. N. Pq., São Paulo, 1957. 58 pp. (mimeographed)

Main themes of this course are: Čech cohomology with coefficients in a sheaf; resolutions; theorem: for a paracompact space the cohomology with coefficients in a sheaf A may be identified with the cohomology of the module of sections of a resolution of A; isomorphism of ordinary Čech cohomology with de Rham-, Alexander-Spanier-, and singular cohomology. The entire exposition is very elegant and straightforward. *W. T. van Est.*

Bokšteĭn, M. F. A new proof of the fundamental theorem of the homologic theory of dimension. Moskov. Gos. Univ. Uč. Zap. **181**. Mat. **8** (1956), 13–44. (Russian)

Elementary proofs, starting from the definitions, are given for these known results for a bicompact space A: For each (finite open) covering w of A there is a w-map of A in the nerve of w. If, for each covering w of A, there is a w-map of A in some bicompact P_w with $\dim P_w \leqq n$, then $\dim A \leqq n$. If T^{q-1} is a subdivision of the boundary S^{q-1} of a q-simplex E^q, any simplicial map $f: T^{q-1} \to S^{q-1}$ of degree zero can be extended to a continuous map $g: E^q \to S^{q-1}$. If the covering dimension $\dim A$ is finite, the integral cohomology dimension of A is equal to $\dim A$. *C. H. Dowker.*

Floyd, E. E. Closed coverings in Čech homology theory. Trans. Amer. Math. Soc. **84** (1957), 319–337.

For each finite closed covering α of a space X, denote by π_α the natural homomorphism of $H_n(X)$ into $H_n(\alpha)$. For a refinement β of α, denote by $\pi_{\beta\alpha}$ the usual homomorphism of $H_n(\beta)$ into $H_n(\alpha)$. If $B \subset A$ are closed sets, denote by $H_n(B; A)$ the image of $H_n(B)$ in $H_n(A)$ under the injection homomorphism induced by inclusion.

If α and β are finite closed coverings of X, and n is a nonnegative integer, β is said to n-refine α $(\beta^n > \alpha)$ if for each element B of β there is an element A of α such that $H_j(B; A)=0$ for all $j \leqq n$. β strongly n-refines α $(\beta^n \gg \alpha)$ if

there is a projection π such that

$$H_j(B^{i_1}\cap\cdots\cap B^{i_n};\pi B^{i_1}\cap\cdots\cap\pi B^{i_n})=0$$

for all $B^{i_1},\cdots,B^{i_n}$ in β and all $j\leqq n$.

If α and β are closed coverings, and if X is compact, α and β are said to determine $H_n(X)$ if $\beta>\alpha$ and if π_α maps $H_n(X)$ isomorphically onto the image of the projection $\pi_{\beta\alpha}$: $H_n(\beta)\to H_n(\alpha)$. The coverings α and β are said to partially determine $H_n(X)$ if the image of π_α equals the image of $\pi_{\beta\alpha}$.

Some work of Kelley and Pitcher [Ann. of Math. (2) **48** (1947), 682–709; MR **9**, 52] on the relationships between the groups of a complex and the groups associated with a covering by subcomplexes is extended to the case of a compact space and a closed covering. Using this, the following theorem is proved: If X is a compact space and if $\alpha_{-1}\ll^n\alpha_0\ll^n\cdots\ll^n\alpha_{2n}$, where each α_i is a closed covering of X, then α_n, α_{2n} determine $H_j(X)$ for all $j\leqq n$, and partially determine $H_{n+1}(X)$.

This theorem is used to give new proofs for three known theorems: the Vietoris mapping theorem, the n-dimensional homology group of an lc^n-space is isomorphic to that of a finite complex, and $H_n(A_i)\cong H_n(A_0)$ for i sufficiently large if $A_i\to A_0$ n-regularly. However, in addition to being new, these proofs have the advantage that they hold not only for the usual case of a field for coefficient groups, but also for compact coefficient groups.

E. G. Begle (New Haven, Conn.).

Simon, Arthur B. n-cyclic elements. I. Duke Math. J. **24** (1957), 1–7.

Let X be a compact Hausdorff space. A closed $T\subset X$ is called a T^n if $H^n(B)=0$ for every closed $B\subset T$ (H^n denotes the nth cohomology group). An n-cyclic element (a K^n) is a (non-degenerate) closed $K\subset X$ such that (1) no T^n disconnects K, (2) K is maximal with respect to (1) [cf. G. T. Whyburn, Amer. J. Math. **56** (1934), 133–146]. Various results concerning T^n- and K^n-sets are proved; it is stated that they are valid, too, for T^n (mod X_0), K^n (mod X_0). The main theorem may be stated as follows: if $B=\{x$: every open $U\supset\{x\}$ intersects infinitely many K^{n-1}-sets$\}$, $\mathfrak{K}'=\{K:K$ is a K^{n-1}, $K\not\subset B\}$, and $K\cap B$ is a T^n whenever $K\in\mathfrak{K}'$, then the kernel of $H^n(X)\to H^n(B)$ is isomorphic with $\sum\{H^n(K):K\in\mathfrak{K}'\}$. The following result is announced without proof: if ind $X\geqq n$, then there is a $K^q\subset X$, $q\geqq n-2$ (ind denotes the inductive dimension). It implies, if ind $X<\infty$, the inequality ind $X\leqq$ cd$X+2$, since, as the author shows, cd$X$ is equal to the largest n such that there is a $K^n\subset X$; cd$X$ denotes here the codimension of X in the sense of H. Cohen [Duke Math. J. **21** (1954), 209–224; MR **16**, 609].

M. Katětov.

Wada, Tôru. Note on a theorem of Hilton. Tôhoku Math. J. (2) **8** (1956), 324–328.

The author announces the theorem: Let $(X;A,B)$ be an excisive triad, $A\cap B=C$, let (A,C) be $(p-1)$-connected ($p\geqq 3$), (B,C) $(q-1)$-connected ($q\geqq 3$) and let C be l-connected ($l\geqq 1$). Then

$$\pi_n(X,C)\cong\pi_n(A,C)+\pi_n(B,C),\quad n<n_0=l+\min(p,q)-1,$$

$$\pi_{n_0}(X,C)\cong\pi_{n_0}(A,C)+\pi_{n_0}(B,C)+\pi_p(A)\otimes\pi_q(B),$$

where the third summand is embedded by way of $\alpha\otimes\beta\to[\alpha,\beta]\in\pi_{n_0}(X;A,B)$. The theorem at least requires restating since $[\alpha,\beta]\in\pi_{p+q-1}(X;A,B)$. In fact the author proves that

$$\pi_n(X,C)\cong\pi_n(A,C)+\pi_n(B,C)+\pi_n(X;A,B),\quad n\leqq n_0.$$

If $l=p+q-1$ he is able to infer from the Blakers-Massey theorem [Ann. of Math. (2) **58** (1953), 409–417; MR **15**, 458] that

$$\pi_{p+q-1}(X)\cong\pi_{p+q-1}(A)+\pi_{p+q-1}(B)+\pi_p(A)\otimes\pi_q(B),$$

the last term being embedded by the (ordinary) Whitehead product.

P. J. Hilton (Manchester).

Iwata, Kôichi. Note on Postnikov invariants of a loop space. Tôhoku Math. J. (2) **8** (1956), 329–332.

The author produces a further proof of the known relation between the Postnikov invariants of a space and those of its loop space [see H. Suzuki, J. Math. Soc. Japan **8** (1956), 93–101; MR **18**, 409].

P. J. Hilton.

★ **Dedecker, Paul. Extension du groupe structural d'un espace fibré.** Colloque de topologie de Strasbourg, 1954–1955, Institut de Mathématique, Université de Strasbourg. 15 pp.

Let E' be a principal fibre space over a base space B with structural group G and let N be a normal subgroup of G. With the natural operations of G on the quotient group $H=G/N$, an associated fibre space E with fibre H is defined. E is also a principal fibre space with structural group H. The extension problem of structural groups is to study whether or not an arbitrarily given principal fibre space E with structural group H can be obtained from a principal fibre space E' in this manner. If G is abelian, N admits a local cross-section in G, and B is a paracompact space, then this extension problem was completely answered in the theory of sheaves. In the present note, the author generalizes this result to the case where N is a subgroup of the center of a (non-abelian) group G, by using low-dimensional cohomology groups with non-abelian coefficients.

S. T. Hu (Detroit, Mich.).

Weier, Josef. Über Abbildungen dreidimensionaler in zweidimensionale Mannigfaltigkeiten. Bayer. Akad. Wiss. Math.-Nat. Kl. S.-B. **1956**, 113–123 (1957).

It is an easy consequence of the Lefschetz coincidence formula that if $f,g:S^n\to S^n$, f essential and g inessential, then f and g have a coincidence; i.e., there is a point $p\in S^n$ such that $f(p)=g(p)$. The author defines a mapping f of a compact 3-manifold P onto a 2-manifold Q to be essential if each f' homotopic to f is an onto mapping. He proves that if $f,g:P\to Q$, f essential and g homotopic to a trivial map ($g\sim g':P\to q\in Q$), then f and g have a coincidence. His proof is based on the lemma that for two such manifolds P and Q there is a positive number α such that if $f,g:P\to Q$ are coincidence-free, there exists a $g'\sim g$ such that $d(f(p),g'(p))>\alpha$ for all $p\in P$.

E. Dyer.

Smale, Stephen. A Vietoris mapping theorem for homotopy. Proc. Amer. Math. Soc. **8** (1957), 604–610.

The following theorem is proved: Let X and Y be connected, locally compact, separable metric spaces, and let X be LC^n. Let f be a mapping of X into Y for which f^{-1} carries compact sets into compact sets. If, for each $y\in Y$, $f^{-1}(y)$ is LC^{n-1} and $(n-1)$-connected, then A) Y is LC^n, and B) the homomorphism of the homotopy groups $\pi_k(X)$ into $\pi_k(Y)$ induced by f is an isomorphism onto for $0\leqq k\leqq n-1$, and is onto for $k=n$.

E. G. Begle.

★**Wilder, R. L. Some mapping theorems with applications to non-locally connected spaces.** Algebraic geometry and topology. A symposium in honor of S. Lefschetz, pp. 378–388. Princeton University Press, Princeton, N. J., 1957. $7.50.

This paper is concerned with extensions, to a wider class of spaces, of two theorems of Lefschetz, of which the first asserts the complex-like nature (i.e., finite Betti numbers) in dimensions $\leq n$ of an lc^n space, and the second is the Lefschetz fixed-point theorem for lc^∞ spaces. The basic tool here is a variety of extensions of the Vietoris mapping theorem, of which the following is a typical example. Let S and T be compact spaces and f a mapping of S onto T. If there is a subset D of T such that both D and $f^{-1}(D)$ are $(n-1)$- and n-acyclic, and if f is n-monotone on $S-f^{-1}(D)$, then $H^n(S)\cong H^n(T)$. Using these, the complex-like structure of a number of non-lc^n spaces is shown by mapping them appropriately onto lc^n spaces. Also, the following fixed-point theorem is proved, where a prime part is either a point at which the space is lc^∞ or a component of the set of points at which the space fails to be lc^∞: Let S be a finite dimensional complex-like space all of whose prime parts are acyclic; let f be a mapping of S into itself which carries prime parts into prime parts and whose Lefschetz number $\theta(f)\neq 0$; then f carries some prime part into itself.

E. G. Begle (New Haven, Conn.).

Vaccaro, Michelangelo. Proprietà topologiche delle rappresentazioni localmente biunivoche. Math. Ann. **133** (1957), 173–184.

A continuous mapping β: $X\to Y$ is a local homeomorphism if each point P of X has a neighborhood that is mapped topologically by β. It is well known that, although a covering mapping is a local homeomorphism, a local homeomorphism need not be a covering mapping. The author proves the following theorem. Suppose that V^n is a differentiable manifold, that a local homeomorphism $\beta: V^n\to W^m$ is differentiable of class C^1 with jacobian matrix everywhere of maximum rank, and that the image $P^n=\beta(V^n)$ of β is a (curved) polyhedron in a differentiable manifold W^m. Then on P^n there can be found a non-trivial (mod 2) n-cycle. In particular, if W^m is the $(n+1)$-dimensional sphere E^{n+1}, the polyhedron P^n separates E^{n+1} into at least two parts.

By contrast the author gives an example of a local homeomorphism β that maps the 2-sphere S^2 into the 3-sphere S^3 in such a way that the image $\beta(S^2)$ is a polyhedron P^2 that does not separate S^3. *R. H. Fox.*

See also: Topological Groups: Taylor. **General Topology:** Egorov; Nagata; Smirnov; Michael; Bing; Harrold. **Geometries, Euclidean and Other:** Fiedler. **Differential Geometry:** Whitney; Thom; Gugenheim and Spencer. **Complex Manifolds:** Spanier. **Algebraic Geometry:** Wallace.

GEOMETRY

Geometries, Euclidean and Other

★**Каган, В. Ф. [Kagan, V. F.] Основания геометрии. Учение об обосновании геометрии в ходе его исторического развития. Часть вторая. Интерпретации геометрии Лобачевского и развитие ее идеи. [Foundations of geometry. A study of foundations of geometry in the course of its historical development. Part II. Interpretations of Lobačevskiĭ's geometry and development of its ideas.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1956. 344 pp. 15 rubles.

The author died in 1953 at the age of 85. The present second part is based on a manuscript found in his papers and was prepared for publication by the members of the seminar on Vector and Tensor Analysis of Moscow University under the direction of G. B. Gurevič, Ya. S. Dubnov, and P. K. Raševskiĭ. It contains a chapter on the axiomatic of a straight line originally planned as the first chapter of the third part (which will not appear) and two appendices by B. A. Rozenfel'd: (1) "The hyperbolic plane as a sphere of imaginary radius" and (2) "Recent investigations in prehistory of non-Euclidean geometry". The rest of the book is a continuation of the history of ideas connected with non-Euclidean geometry. Chapter 11 deals with realizations of Lobačevskiĭ's geometry on surfaces of constant curvature. Chapter 12 discusses the role of the idea of motion in the foundations of geometry and contains an elementary but detailed exposition of the theory of groups of transformations. Chapter 13, starting with a mapping of the hyperbolic plane on the interior of the unit circle that goes back to Beltrami, leads to Klein's interpretation in terms of projective geometry. It also contains a section on geometrical constructions in the Lobačevskiĭ plane in terms of Klein's model. Chapter 14, "Second order curves in the Lobačevskiĭ geometry" is based on the work of the author's students M. G. Freĭdin and V. G. Yampol'skiĭ, and contains a complete classification of these curves. Chapter 15, "Development of Klein's ideas", brings the introduction and development of the work of Cayley, of Riemann, and of Helmholtz. Chapter 16 is devoted to the foundations of elliptic geometry on a straight line, in the plane and in three dimensions; a special section treats Clifford's lines and surfaces. Chapter 17 discusses the nine species of non-Euclidean geometry of two dimensions in terms of models built in the Euclidean plane. Chapter 18 deals with Poincaré's interpretation of the geometry of Lobačevskiĭ. The style of the exposition throughout is the same as in the first part of the book [1949; MR **12**, 731], lucid and explicit.

Chapter 19, "Axiomatic of a Straight Line" (mentioned above), contains the beginning of a systematic presentation of the foundations of geometry. It includes a set of axioms for the concept of magnitude (due to S. O. Šatunovskiĭ) together with a discussion of its completeness and independence. *G. Y. Rainich* (Ann Arbor, Mich.).

Fiedler, Miroslav. Einige Sätze aus der metrischen Geometrie der Simplexe in euklidischen Räumen. Schr. Forschungsinst. Math. **1** (1957), 157.

The author refers to an earlier (Czech) paper [Časopis Pěst. Mat. **79** (1954), 297–320; MR **16**, 1045] for a discussion of the problem of constructing an n-simplex in which each angle between $(n-1)$-dimensional sides is specified as acute, right or obtuse. It appears that the problem is solvable if and only if the vertices of the simplex, together with the edges which are to cross acute angles, constitute a connected graph. Here the author notes some simple geometrical properties of a simplex for which this graph is a simple arc. *W. T. Tutte.*

Lorent, H. Courbes associées à deux courbes données dépendant chacune d'un seul paramètre. Bull. Soc. Roy. Sci. Liège **25** (1956), 554–569.

Posthumous note consisting of analytic geometry

exercises, for specific cases, on the locus of intersection of two curves, depending on parameters a and b respectively, when a and b are in a given algebraic relation.

Szász, Pál. On hyperbolic trigonometry interpreted as the Poincaré circle model. Magyar Tud. Akad. Mat. Fiz. Tud. Oszt. Közl. **6** (1956), 73–80. (Hungarian)

Bushko-Zhuk, M. M. On transformations of inversion. Ukrain. Mat. Ž. **9** (1957), 101–104. (Russian. English summary)

Hyperbolic planes, euclidean planes and spheres are realized by equidistant surfaces, limit spheres, spheres in a hyperbolic space H^3. Let S be one of these surfaces in H^3 and C a curve of constant curvature on S. The lines of H^3 tangent to S and orthogonal to C belong to a bundle. Projection along the lines of this bundle defines a mapping of S on itself which is called an inversion. Inversions are conformal, and curves of constant curvature passing through corresponding points are orthogonal to C. An analytic expression for inversion is derived in the hyperbolic case as an example. Inversions are characterized axiomatically in terms of the intrinsic geometry on S. The spherical, euclidean and hyperbolic geometries are, of course, characterized by the absence of points without image, uniqueness of such a point, and the existence of a domain of such points, respectively. *H. Busemann.*

★ **Coxeter, H. S. M. Regular honeycombs in hyperbolic space.** Proceedings of the International Congress of Mathematicians, 1954, Amsterdam, vol. III, pp. 155–169. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam, 1956. $7.00.

The regular honeycombs of hyperbolic space are partitions of hyperbolic S_r into congruent regular polytopes analogous to the partition of a sphere (regarded as elliptic space) into the central projections of the faces of an inscribed polytope, or of the Euclidean plane into triangles, squares, or hexagons, or of Euclidean S_3 into cubes. Standard notation is defined by induction: $\{m\}$ is the regular m-gon, $\left\{\frac{m}{p}\right\}$ a star m-gon whose perimeter goes p times round the center; $\{m_1, \cdots, m_r\}$ is an $r+1$ dimensional polytope or r dimensional honeycomb with cell $\{m_1, \cdots, m_{r-1}\}$ and vertex figure $\{m_2, \cdots, m_r\}$.

In the hyperbolic plane there are the honeycombs $\{p, q\}$, $(p-2)(q-2)>4$ (without this inequality $\{p, q\}$ is a regular polyhedron or a Euclidean honeycomb) including $\{\infty, q\}$†, $\{p, \infty\}$*; (stellated) $\left\{m, \frac{m}{2}\right\}$ and $\left\{\frac{m}{2}, m\right\}$, $m>6$, of density 2.

In three dimensions $\{3, 5, 3\}$, $\{4, 3, 5\}$, $\{5, 3, 4\}$, $\{5, 3, 5\}$, $\{3, 4, 4\}$*, $\{3, 3, 6\}$*, $\{4, 3, 6\}$*, $\{5, 3, 6\}$*, $\{4, 4, 3\}$†, $\{6, 3, 3\}$† $\{6,3,4\}$†, $\{6, 3, 5\}$†, $\{6, 3, 6\}$*†, $\{4, 4, 4\}$*†, $\{3, 6, 3\}$*†; (none stellated).

In four dimensions $\{3, 3, 3, 5\}$, $\{4, 3, 3, 5\}$, $\{5, 3, 3, 5\}$, $\{5, 3, 3, 4\}$, $\{5, 3, 3, 3\}$, $\{3, 4, 3, 4\}$*, $\{4, 3, 4, 3\}$†; (stellated) $\{\frac{5}{2}, 5, 3, 3\}$, $\{3, 3, 5, \frac{5}{2}\}$ of density 5, and $\{3, 5, \frac{5}{2}, 5\}$, $\{5, \frac{5}{2}, 5, 3\}$ of density 10.

In five dimensions $\{3, 4, 3, 3, 3\}$*, $\{3, 3, 3, 4, 3\}$†, $\{3, 4, 3, 3, 4\}$*†, $\{4, 3, 3, 4, 3\}$*†, $\{3, 3, 4, 3, 3\}$*†; (none stellated).

Here * indicates that the vertices are all at infinity, † that the cells are inscribed in horospheres, so that their centers are all at infinity. In either case one vertex or, if both are combined, two vertices of the fundamental region are at infinity. If the fundamental region is allowed to extend "beyond infinity" there are of course further possibilities, but the problem becomes too vague to be of much interest. *P. Du Val* (London).

Gruenbaum, B. A proof of Vazonyi's conjecture. Bull. Res. Council Israel. Sect. A. **6** (1956), 77–78.

Heppes, A. Beweis einer Vermutung von A. Vázsonyi. Acta Math. Acad. Sci. Hungar. **7** (1956), 463–466. (Russian summary)

Straszewicz, S. Sur un problème géométrique de P. Erdös. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 39–40, IV–V. (Russian summary)

The authors prove independently the following conjecture of Vázsonyi: Let there be given n points $x_1, x_2, \cdots, x_n$ in three-dimensional space. Assume that the diameter of this point set is 1; then there are at most $2n-2$ pairs (x_i, x_j) whose distance is 1. The method of proof of all three papers is similar and uses Euler's relation on polyhedra. *P. Erdös* (Haifa).

Nottrot, R. Note on a previous paper on torsion. Nederl. Akad. Wetensch. Proc. Ser. A. **60**=Indag. Math. **19** (1957), 159–161.

This note compares the quasi-metric notion of torsion introduced in a previous paper [Haantjes and Nottrot, same Proc. **58** (1955), 405–410; MR **17**, 524] with the purely metric notions of torsion due (independently) to Alexits [Compositio Math. **6** (1939), 471–477] and the reviewer [Theory and applications of distance geometry, Oxford, 1953; MR **14**, 1009]. It is shown that the former notion yields the classical torsion for curves of a Riemannian space, while the latter notions do not.

L. M. Blumenthal (Columbia, Mo.).

Alexandrow, A. D. Über eine Verallgemeinerung der Riemannschen Geometrie. Schr. Forschungsinst. Math. **1** (1957), 33–84.

By K-plane denote for $K=0$ the euclidean, for $K<0$ the hyperbolic plane of curvature K, and for $K>0$ an open hemisphere of curvature K. A triangle $\Delta=abc$ in metric space is the union of segments $T(a, b)$, $T(b, c)$ and $T(c, a)$ (a segment $T(x, y)$ is a curve of length xy from x to y). With Δ we associate an isometric triangle $(abc)^K=\Delta^K$ in a K-plane, where (*) $ab+bc+ca<2\pi K^{-\frac{1}{2}}$ is assumed for $K>0$. If M, N are segments issuing from p, $x \in M-p$, $y \in N-p$, and $\gamma_{MN}{}^K(x, y)$ is the angle at p in $(pxy)^K$, then $\alpha(M, N)=\lim\sup_{x\to p, y\to p} \gamma_{MN}{}^K(x, y)$ is the upper angle between M and N at p. The defect $\delta^K(\Delta)$ relative to K of Δ is the sum of the upper angles of Δ at a, b, c minus the angle sum in $(abc)^K$. A space R_K with curvature $\leqq K$ is a metric space in which a $T(x, y)$ exists for given x, y, $\delta^K(\Delta)\leqq 0$ for any triangle Δ, and (*) holds when $K>0$. The condition $\delta^K(\Delta)\leqq 0$ is equivalent to the condition $\limsup \delta^0(a_\nu b_\nu c_\nu)/A^0(a_\nu b_\nu c_\nu)\leqq K$ for any sequences of triangles $a_\nu b_\nu c_\nu$ tending to a point, where $A^0(a_\nu b_\nu c_\nu)$ is the area of $(a_\nu b_\nu c_\nu)^0$.

In an R_K, the angle $\gamma_{MN}{}^K(x, y)$ above is a non-decreasing function of px and py, so that $\lim_{x\to p, y\to p} \gamma_{MN}{}^K(x, y)$ exists, and the angles in abc are not greater than the corresponding angles in $(abc)^K$. The distance of the midpoints of two sides in (abc) equals at most the distance of the corresponding midpoints in $(abc)^K$. The segment $T(x, y)$ is unique for any x, y, and the spheres (of radius $\leqq \pi/2K^{\frac{1}{2}}$ if $K>0$) are convex. In any metric space in

which $T(x, y)$ exists for any x, y and depends continuously on x, y, the condition $\delta^K(abc) \leqq 0$ required locally implies $\delta^K(abc) \leqq 0$ for arbitrary abc. If a point in an R_K has a neighborhood N homeomorphic to E^n and $\delta^K(\Delta) = 0$ for all ΔCN, then N is isometric to a domain in a space with constant curvature K. With a proper definition of area for continuous images of a circular disk it is shown that any closed curve of length L in an R_K spans a surface whose area is at most that of a circle of length L in a K-plane. If C is a curve of length L in an R_K connecting a and b, and $h = \sup_{x \in C} xT(a, b)$, then h is at most equal to the length of the altitude of a triangle in a K-plane whose base has length ab and whose other sides have length $L/2$. *H. Busemann* (Los Angeles, Calif.).

Busemann, Herbert. Metrizations of projective spaces. Proc. Amer. Math. Soc. **8** (1957), 387–390.

Theorem I. Let the projective space P^n $(n \geqq 2)$ be metrized as a G-space [cf. Busemann, The geometry of geodesics, Academic Press, New York, 1955; MR **17**, 779] such that the geodesics are the projective lines. Then P^n can be imbedded in P^{n+1} such that the metric in P^n is preserved and the geodesics in P^{n+1} are the projective lines.

Theorem II. In P^2 let a system Σ' of curves be given such that each curve in Σ' is a closed Jordan curve and two distinct points of P^2 lie on exactly one curve in Σ'. Then P^2 can be metrized as a G-space such that the curves in Σ' become the geodesics.

Both of these problems were solved by the author (loc. cit.) with E^2 in place of P^n, P^2 respectively. Skornyakov [Izv. Akad. Nauk SSSR. Ser. Mat. **19** (1955), 471–482; MR **17**, 522] proved Theorem II (an inverse problem in the calculus of variations in the large). By modifying Skornyakov's method the author provides simultaneous proofs of both theorems, considerably shortening the argument necessary for Theorem II in the process. *L. W. Green* (Minneapolis, Minn.).

Bereis, R.; und Brauner, H. Die automorphen involutorischen Korrelationen koaxialer projektiver Schraubungen. Österreich. Akad. Wiss. Math.-Nat. Kl. S.-B. II. **165** (1956), 327–355.

Verf. betrachten die ∞^3 Schraublinien und Schraubtorsen von ∞^1 koaxialennichteuklidischen Schraubungen und fragen nach solchen involutorischen Korrelationen, welche die Gesamtheit der Schraublinien in die der Torsen überführen. Es gibt neun Typen solcher Korrelationen. Ihre Zusammensetzung mit jenem kubischen Nullsystem das jeder Bahnkurve einer Schraubung ihre Torse zuweist liefert kubische Punkt- und Ebenenverwandtschaften die die Gesamtheit koaxialer Schraubungen in sich überführen. Die Untersuchung ist analytisch und wird allgemein projektiv so geführt dass man leicht auf die einzelnen nichteuklidischen Metriken spezialisieren kann. Auch der Grenzfall der euklidischen Metrik wird betrachtet. *O. Bottema* (Delft).

★ **Haack, Wolfgang. Darstellende Geometrie. III. Axonometrie und Perspektive.** Sammlung Göschen Band 144. Walter de Gruyter and Co., Berlin, 1957. 127 pp. DM 2.40.

Volumes I and II (1954) were reviewed in MR **15**, 980, 981. This third volume opens with an introduction into orthogonal axonometry with special attention to the ratio $\frac{1}{2}:1:1$. There is also a section on oblique axonometry and the theorem of Pohlke, which is proved by showing that there exists a plane intersecting a given triangular prism in a triangle similar to a given triangle. Then follow the principles of free and applied perspective with applications to circles and spheres. There are constructions for the case that points lie outside the frame of drawing, and for the shadows of solids both in axonometry and perspective. Despite the small size of the book the figures are remarkably clear. *D. J. Struik.*

See also: Linear Algebra: Motzkin and Taussky. **Groups and Generalizations:** Coxeter; Shephard. **Topological Groups:** Freudenthal; Tits. **Algebraic Geometry:** Germer.

Convex Domains, Integral Geometry

See: Geometries, Euclidean and Other: Gruenbaum. **Differential Geometry:** Pogorelow. **Programming, Resource Allocation, Games:** Fan, Glicksberg and Hoffman.

Differential Geometry

Borisov, Yu. F. An extrinsic property of geodesics on smooth surfaces. Vestnik Leningrad. Univ. **11** (1956), no. 19, 35–40. (Russian)

Let S be a surface of class C' in E^3. A polygon with vertices $q_1, \cdots, q_n$ on S is called normal if the (or a suitable) plane through q_{i-1}, q_i, q_{i+1} $(i = 2, \cdots, n-1)$ contains the normal to S at q_i. For the points p, p' on S let $UCVCS$, where V is homeomorphic to a closed disk, and U is open and contains the closed disk about p (in the sense of the intrinsic metric on S) with the intrinsic distance of p and p' as radius. Then there exists a shortest connection of p and p' on S which is the limit of normal polygons $q_1, \cdots, q_n$ with $q_1 = p$, $q_n = p'$. Whether this holds for every shortest connection of p and p' is not known. *H. Busemann* (Los Angeles).

★ **Pogorelow, A. W. Die Verbiegung konvexer Flächen.** Schriftenreihe des Forschungsinstituts für Mathematik. Heft 5. Akademie-Verlag, Berlin, 1957. 135 pp. DM 18.50.

This is a translation by A. Lingener of the Russian work reviewed in MR **14**, 400.

Flett, T. M. The definition of a tangent to a curve. Edinburgh Math. Notes no. **41** (1957), 1–9.

It is shown that a curve with a continuous tangent, as defined in elementary geometry, is not necessarily rectifiable and regular if parametrized to arc-length. Three alternative definitions of a tangent are given, which are called α, β and γ tangent. The author examines whether the existence of one of them includes the existence of the others and proves that a curve with an α tangent at each point is regular and rectifiable. *O. Bottema.*

Mokriščev, K. K. On curves on a surface, whose normal curvature, geodesic curvature and torsion are connected by an equation. Rostov. Gos. Ped. Inst. Uč. Zap. no. 3 (1955), 95–102. (Russian)

The equation of the title is first assumed to be linear and the class of curves is found to include those investi-

gated by N. G. Tuganov [C. R. (Dokl.) Acad. Sci. URSS (N.S.) **30** (1941), 383–385; MR **3**, 17]. The author also examines quadratic relations (which include curves on developable surfaces). *L. W. Green.*

Černyaev, M. P. The deviation axis (affine normal). Rostov. Gos. Ped. Inst. Uč. Zap. no. 3 (1955), 135–150. (Russian)

By using non-parametric, non-affine-invariant formulae, the author obtains analytically several theorems from Chapter I of Blaschke [Vorlesungen über Differentialgeometrie, Bd. II, Springer, Berlin, 1923]. The affine analogue of a theorem of d'Occagne [Bull. Soc. Math. France **20** (1892), 49–59] is also obtained. *L. W. Green.*

Drîmbă, Constanţa; et Drîmbă, Constantin. Formules intégrales ayant trait à des longueurs, aires et volumes. Com. Acad. R. P. Romîne. **6** (1956), 387–391. (Romanian. Russian and French summaries)

The idea of the paper is suggested by the observation that the derivative with respect to the radius of the area of a circle is the length of the circumference. The paper deals with a closed curve F and a family of curves filling the interior of F such that through every point (except one) of the interior passes one curve of that family; thus to every point x, y of the interior is assigned a number $K(x, y)$, the curvature at x, y of the curve of the family that passes through this point. The length L of the given curve is found to be the integral of K extended over the interior of the curve. For a closed surface in space an analogous formula is obtained in which the mean curvature plays the part that K plays above.

G. Y. Rainich (Ann Arbor, Mich.).

Mishra, R. S.; and Upadhyay, M. D. Spherical representation of a congruence. Ganita **7** (1956), 55–64.

In this paper the authors consider the following situation: a surface in three-dimensional space given by $x^i=x^i(u^\alpha)$ ($i=1, 2, 3$ and $\alpha=1, 2$) together with a field of directed lines, one at each point of the surface, these being given by their direction cosines, $\lambda^i(u^\alpha)$; in brief, a rectilinear congruence. All considerations are purely local. The situation just described leads in a natural way to a representation of the surface on the unit sphere: to the point with parameters (u^α) corresponds the point $\lambda^i(u^\alpha)$. When the field of lines is normal to the surface, this gives the usual spherical representation. The paper is then devoted to finding expressions for the Christoffel symbols of the first and second kind for the spherical representation of the surface in terms of the fundamental tensors of the surface itself. *W. Boothby* (Evanston, Ill.).

Coz, Marcel. Sur les cas riemanniens de métriques variationnelles $\delta s=f(u, v; du, dv)$. C. R. Acad. Sci. Paris **244** (1957), 1873–1875.

The author assumes curvilinear coordinates (u, v, w) to be introduced in 3-dimensional Euclidean space by $x=x(u, v, w)$, $y=y(u, v, w)$ and $z=z(u, v, w)$, and he assumes the Euclidean metric to be given in these coordinates by

$$ds^2=Adu^2+2Bdudv+Cdv^2+2dw(Fdu+Gdv)+Hdw^2.$$

Then fixing w_1 and w_2, $w_1<w_2$, he introduces on each surface $w=k$, $w_1\leqq k\leqq w_2$, a metric

$$\delta s^2=\Big[\int_{w_1}^{w_2}[H(Adu^2+2Bdudv+Cdv^2)-(Fdu+Gdv)^2]^{\frac{1}{2}}dw\Big]^2$$

(this metric assigns as length to each curve Γ on $w=k$ the area of the cylinder generated by all $(u(t), v(t), w)$ with $w_0\leqq w\leqq w_1$ and $(u(t), v(t), k)$ describing Γ), and he asks under what conditions δs^2 is Riemannian. It is shown that this happens if and only if the mapping between the surfaces $w=k$ for various k, given by allowing points with the same u, v coordinates to correspond, is conformal and the equation $Fdu+Gdv+Hdw=0$ is integrable. This condition is then used to determine all such metrics, and to show that the surfaces $w=$const. must be planes or concentric spheres. *W. Boothby* (Evanston, Ill.).

Gorowara, K. K. On certain ruled surfaces. Ganita **5** (1954), 105–112 (1955).

L'A. dimostra alcune proprietà metriche di una superficie rigata S. Nella prima parte egli considera il triedro di Blaschke (A_1, A_2, A_3), costituito dalla generatrice A_1, dalla normale A_2 nel punto centrale M di A_1 e dalla perpendicolare A_3 ad A_1 ed A_2, e dimostra che: La superficie S e la rigata generata dal'asse istantaneo di rotazione $A_1^{(1)}$ del triedro (A_1, A_2, A_3) hanno la stessa linea di strizione se e solo se questa è un'asintotica.

In tal caso $A_1^{(1)}$ genera una sviluppabile.

Nella seconda parte l'A. considera il triedro geodetico associato ai punti di una curva C di S e studia le rigate generate dagli spigoli del triedro in relazione alla S.

C. Longo (Parma).

Miron, R. Sur la sphère non holonome et le plan non holonome. An. Şti. Univ. "Al. I. Cuza" Iaşi. Secţ. I. (N.S.) **1** (1955), 43–52. (Romanian. Russian and French summaries)

Given a non-holonomic manifold V_3^2 in the euclidean 3-space S_3, one defines the 1st and 2nd fundamental forms, Φ and Ψ, as in the classical theory of surfaces in S_3. A V_3^2 is a non-holonomic sphere Σ_3^2 if, in every point, Ψ is proportional to Φ. A V_3^2 is a non-holonomic plane Π_3^2 if Ψ is identically equal to zero.

In this paper some properties of Σ_3^2 and Π_3^2 are derived dealing with the various types of curvature and torsion which can be defined in a V_3^2, and with lines of curvature, asymptotic lines, etc. In addition, the author introduces for a V_3^2 a Dupin indicatrix for the curvatures and a similar indicatrix named "Bonnet" for the torsions.

R. Blum (Saskatoon, Sask.).

Miron, R. Sur la réduction à une forme canonique du groupe intrinsèque d'un espace non holonome. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **8** (1956), 631–645. (Romanian. Russian and French summaries)

The author shows how a method introduced by M. Haimovici [C. R. Acad. Sci. Paris **210** (1940), 595–596; MR **2**, 163] can be applied to reduce the rigid group of an anholonomic space V_n^m to a canonical form. It is assumed that the Pfaff system defining the V_n^m has three derived systems and that the ranks of three matrices have certain values. The notation used is that of the calculus of congruences of G. Vranceanu. *R. Blum.*

Urban, A. A second form of curvature of a surface in R_3. Časopis Pěst. Mat. **78** (1953), 73–88. (Czech)

Toutes les courbes de la surface S qui touchent le vecteur i^a ($a=1, 2$) en point non-singulier A possèdent aussi

le vecteur

$$v^{\kappa}_{n}=i^{a_0}\cdots i^{a_n}H^{\cdot\cdots\cdot}_{a_0\ldots a_n\kappa}$$

$$(n=0,1,2,\cdots;\ a_0,a_1,\cdots=1,2;\ \kappa=1,2,3),$$

où $H^{\cdot\cdots\cdot}_{a_0\ldots a_n\kappa}=D_{a_n}\cdots D_{a_1}B_{a_0\kappa}$, et $D_a v^b=\nabla_a v^b$ pour le vecteur de la surface, $D_a v^\kappa=B_a{}^\mu\partial_\mu v^\kappa$ pour le vecteur de l'éspace R_3. Le vecteur v^κ_n est le n-ième vecteur de la courbure adjoint au vecteur tangent i^a. Dans le travail on étudie le 2d vecteur de courbure pour lequel

$$v^{\kappa}_{2}=-\frac{1}{R^2}i^\kappa-\frac{\varepsilon}{RT}j^\kappa+\frac{1}{s}N^\kappa,$$

où $\varepsilon=[i^\kappa, j^\kappa, N^\kappa]$, $1/R$ est la courbure géodésique, $1/T$ est la torsion géodésique adjointe au vecteur i^κ, $1/s$ est le scalaire de Codazzi, j^κ est le vecteur normal de la courbe, N^κ est le vecteur normal de la surface. On montre les propriétés du vecteur v^κ_2 et on trouve le lieu des points limites M du vecteur v^κ_2 $(=AM)$ si l'on change le vecteur i^{a_1} en point A. On généralise aussi la notion des directions asymptotiques à l'aide du premier vecteur v^κ_1 des asymptotiques qui est situé dans le plan tangent de la surface.

F. Vyčichlo (Prague).

Šulikovskiĭ, V. I. Tensor theoretical presentation of the method of the movable trihedron of a surface in E_3. Uč. Zap. Kazan. Univ. **115** (1955), no. 14, 61–68. (Russian)

Zaharov, D. A. Isotopy of paraboloids. I. Ivanov. Gos. Ped. Inst. Uč. Zap. Fiz.-Mat. Nauki **10** (1956), 19–44. (Russian)

Two sets M, M' in a metric space R are equimorph if there exists a topological mapping of R on itself which together with its inverse is uniformly continuous and takes M into M'. If the mapping and its inverse satisfy Lipschitz conditions, then M and M' are isotopic. Consider paraboloids of the form $z=f(x_1,\cdots,x_k)+g(x_1,\cdots,x_k)$ in E^{k+1}, where f is a positive definite form of even degree n and g is a polynomial of degree less than n. Two such paraboloids are isotopic if the degrees of f are the same, but not even equimorph if these degrees are different.

In the case of two variables, a paraboloid $z=f(x_1,x_2)$, where f is a not necessarily definite form, can, by an affine transformation and introduction of polar coordinates r, φ in the (x_1,x_2)-plane, be put in the canonical form

$$z=r^n\prod_{i=1}^{k-1}\sin^{p_i}\left(\varphi-\frac{(i-1)\pi}{k}\right)\qquad(\Sigma\, p_i\leqq n;\ r=(x_1^2+x_2^2)^{\frac{1}{2}}).$$

Therefore there is only a finite number of isotopy classes for fixed n. With their intrinsic metrics only the paraboloids with the canonical form $z=r^n\sin^k\varphi$ are isotopic to the plane. Finally, the volume invariants of Efremovič for paraboloids are investigated. Some of the proofs are quite involved. *H. Busemann* (Los Angeles, Calif.).

Strel'cov, V. V. On a certain mapping of a surface of negative curvature. Izv. Akad. Nauk. Kazah. SSR. Ser. Mat. Meh. **1956**, no. 5(9), 29–44. (Russian)

All concepts are understood in the general sense of A. D. Alexandrov; in particular "swerve" will correspond to "total geodesic curvature" in the classical case. Let S be a surface in E^3 homeomorphic to a closed disk. Let A be a subarc of the boundary of S such that the positive part of its swerve towards S does not surpass π. Then S can be mapped in E^2 so that A is mapped on an arc A_0 of the same length and every subarc of A_0 has a non-negative swerve towards the image S_0 of S, which equals the positive part of the swerve of its pre-image. Finally, and principally, an arbitrary curve C in S goes into a curve in S_0 whose length does not exceed that of C.

H. Busemann (Los Angeles, Calif.).

Fréchet, Maurice. Détermination des surfaces minima du type $a(x)+b(y)=c(z)$. C. R. Acad. Sci. Paris **244** (1957), 145–147.

The author determines minimal surfaces of the form $c(z)=a(x)+b(y)$. These are all expressible in terms of elementary and elliptic functions. Proofs appear in the paper reviewed below. *H. L. Royden.*

Fréchet, Maurice. Détermination des surfaces minima du type. $a(x)+b(y)=c(z)$. Rend. Circ. Mat. Palermo (2) **5** (1956), 238–259 (1957). (Esperanto summary).

Diese Arbeit enthält die Beweise zu der in der oben referierten Arbeit gegebenen Aufstellung aller Minimalflächen des im Titel genannten Typs, die hinreichend differenzierbar sind, um das hier verwendete formale Verfahren zu ermöglichen. Für die Komponenten einer Parameterdarstellung der Fläche durch $x=A(u)$, $y=B(v)$, $z=C(u+v)$ erhält man eine Funktionalgleichung, die durch Eliminationen und Quadraturen gelöst wird.

K. Krickeberg (Würzburg).

Šul'man, T. A. Triply conjugate systems of surfaces R. Asymptotic transformations of systems R. Stratifiable complexes R. Mat. Sb. N.S. **39(81)** (1956), 293–314. (Russian)

On considère trois systèmes (à un paramètre) des surfaces avec les propriétés suivantes: a) Les surfaces d'un système sont les surfaces R, et b) les surfaces des autres deux systèmes coupent chaque surface R considérée dans le réseau R. On appelle ces systèmes les R-systèmes. L'auteur introduit la correspondance asymptotique entre deux R-systèmes de la manière suivante: a) Aux réseaux R des courbes conjuguées sur les surfaces d'un R-système correspondent les réseaux R sur les surfaces du second R-système, et b) à chaque surface R d'un R-système correspond la surface R du second R-système dans la correspondance asymptotique, c'est-à-dire les deux surfaces qui sont en correspondance sont les surfaces focales de la congruence W.

Dans le travail l'auteur a résolu le problème d'existence d'un couple de R-systèmes qui sont dans la correspondance asymptotique et a donné la charactéristique géométrique de ces systèmes. La méthode des preuves est basée sur la méthode des formes différentielles extérieures de E. Cartan. Le théorème fondamental du travail montre: Il existe un couple de R-systèmes qui sont dans la correspondance asymptotique dépendant de treize fonctions arbitraires d'un variable. *F. Vyčichlo* (Prague).

Barner, Martin. Über konforme, kreistreue Abbildung von Kreisflächen. Arch. Math. **8** (1957), 66–76.

A circular surface in E^3 is generated by a one-parameter family of circles. Two such surfaces belong to the same class if there is a mapping of one on the other which is conformal and such that generators are mapped into generators under preservation of cross-ratio. It is shown that there is a four-parameter family of classes of circular

surfaces. The surfaces in one class are either such that their generators touch real (possibly degenerate) curves, in which case two surfaces in the class can be continuously deformed into each other within the class; or else the class splits into two sub-classes within either of which such deformations exist but no deformation exists for surfaces in different sub-classes. *H. Busemann.*

Vekua, I. N. Sur certaines conditions de la non-déformabilité des surfaces à courbure positive. Czechoslovak Math. J. **6(81)** (1956), 143–160. (Russian. French summary)

On montre dans ce travail quelques nouveaux critères de non-déformabilité des surfaces à courbure de Gauss positive dans le cas de déformations infiniment petites. Ces problèmes sont étroitement liés à des problèmes de Mécanique conduisant à un système d'équations aux dérivées partielles de la forme

$$(1)\quad \frac{\partial\varphi}{\partial x}-\frac{\partial\psi}{\partial y}+a\varphi+b\psi=0,\quad \frac{\partial\varphi}{\partial y}+\frac{\partial\psi}{\partial x}+c\varphi+d\psi=0$$

que l'on peut transformer en

$$(2)\quad \frac{\partial w}{\partial\bar z}+B(z)\bar w=0;\ w=\varphi+i\psi,\ \frac{\partial}{\partial\bar z}\equiv\tfrac12\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right).$$

Certaines propriétés des fonctions analytiques qui vérifient le système (2) pour $B(z)=0$ se transportent au cas des fonctions analytiques généralisées, c'est-à-dire de solutions généralisées (dans le sens de M. S. L. Sobolev) du système (2); par le procédé de géométrisation on obtient certains résultats concernant la non-déformabilité des surfaces à courbure positive.

Ensuite on cite (également sans démonstration) certains résultats qui ont été publiés par T. Carleman [C. R. Acad. Sci. Paris **197** (1933), 471–474], concernant l'existence des solutions du système (1) à la condition de frontière

$$(3)\qquad \alpha\varphi+\beta\psi=\gamma.$$

Ces résultats donnent certains théorèmes sur la non-déformabilité des ovaloïdes (c'est-à-dire des surfaces fermées à courbure positive) soumis à des liaisons conduisant à des conditions de frontière du type Hilbert (3).

Author's summary.

Rembs, Eduard. Infinitesimale Verbiegungen mit Randvorgabe des Geschwindigkeitsvektors oder des Drehvektors. Arch. Math. **8** (1957), 77–80.

Let $x(u, v)$ be a (sufficiently differentiable) vector-function which is thought of as yielding a representation of a surface in Euclidean three-space. An infinitesimal deformation of $x(u, v)$ is determined by a vector-function $z(u, v)$ satisfying the condition $dx\,dz=0$. This vector-function $z(u, v)$, termed the velocity vector of the infinitesimal deformation, determines through the equation $dz=y\times dx$ a vector-function $y(u, v)$, the rotation vector of the infinitesimal deformation. Generalizing previous investigations, the author raises the question as to what extent the vectors z and y can be prescribed on the boundary of the surface, and what inferences may be drawn (from the behavior of z and y on the boundary) concerning the corresponding infinitesimal deformation. Special instances include conditions upon the curvature and torsion of the boundary, as well as cases of fixed and "sliding" boundaries. The somewhat involved general results are obtained by reducing the geometrical issues to general theorems on differential equations. *T. Radó.*

Eriksson, Folke. An estimate of Fréchet distances on surfaces of bounded curvature. Math. Scand. **4** (1956), 309–327.

This paper deals with issues belonging to the intrinsic geometry of surfaces, and hence the author uses the term "surface" to refer to a line element $ds^2=Edx^2+2Fdxdy+Gdy^2$ given in a certain region in the xy-plane. Given two sets A, B of the surface, their geodesic distance $d(A, B)$ is defined as the greatest lower bound of the lengths (in terms of the given line element ds^2) of all those curves which connect a point of A with a point of B. The Fréchet distance $D(A, B)$ of A and B is defined by the formula

$$D(A,B)=\max[\text{l.u.b.}_{b\in B}\, d(A,b),\ \text{l.u.b.}_{a\in A}\, d(a,B)].$$

In what follows, a and b denote points of the surface, L is the geodesic distance of a and b, c_1 and c_2 are curves connecting a and b of lengths L_1, L_2 (in terms of the given line element ds^2), D is the Fréchet distance of c_1 and c_2, and K is the Gauss curvature of the surface. For the case $K\leqq 0$ Beurling established the inequality

$$L^2+D^2\leqq(L_1+L_2)^2/4.$$

The purpose of the present paper is to establish corresponding inequalities for the case when the assumption $K\leqq 0$ is replaced by the assumption $K\leqq K_0$, where K_0 is an arbitrary constant. The author assumes first that there exists a geodesic connecting the points a and b, and that the line element ds^2 is sufficiently regular to permit the use of the formal apparatus of classical differential geometry. He then derives the following inequalities. Let $2k$ be the positive square root of K_0 if $K_0>0$, and let $2c$ be the positive square root of $-K_0$ if $K_0<0$. Then the inequalities in question are, for the cases $K_0>0$ and $K_0<0$, respectively,

$$\cos kL\cos kD\geqq\cos\tfrac12 k(L_1+L_2),$$

provided that $2kL<\pi$, $2kL_1\leqq\pi$, $2kL_2\leqq\pi$, and

$$(1)\qquad \cosh cL+\cosh cD\leqq\cosh\tfrac12 c(L_1+L_2).$$

For the case $K_0<0$, the same inequality is then shown to hold under less restrictive assumptions. It is known that the class of surfaces with $K\leqq 0$ is contained in the class of surfaces with a line element of the form

$$(2)\qquad ds^2=e^{2u(x,y)}(dx^2+dy^2),$$

where $u(x, y)$ is an arbitrary subharmonic function. The author notes that a similar statement holds for surfaces with $K\leqq K_0$, where subharmonic functions are replaced by "functions of curvature $\leqq K_0$". The twice differentiable functions of this class coincide with those functions $u(x, y)$ which satisfy the differential inequality $\Delta u\geqq -K_0e^{2u}$. For the case $K_0<0$ the author proves that the inequality (1) remains valid under the only assumption that the function u appearing in (2) is a "function of curvature $\leqq K_0$". *T. Radó* (Columbus, Ohio).

Kunle, Heinz. Über T-Figuren in einem quadratischen Komplex. Math. Z. **64** (1956), 270–285.

Etant données quatre congruences de droites, elles forment un système appelé 'T-Figur', si les quatre droites appartenant chacune à l'une des quatre congruences et correspondant à un même système de valeurs u, v des paramètres constituent un quadrilatère gauche dont les sommets décrivent respectivement les surfaces focales des congruences correspondantes. L'auteur étudie les 'T-Figuren' formés des congruences de droites apparte-

nant toutes à un même complexe quadratique de droites, et montre entre autres, selon le type du complexe quadratique envisagé, que: si le complexe quadratique fait partie du type 46, défini dans Enzycl. Math. Wiss., Bd. III, T. 2, pp. 1126–1133 [Teubner, Leipzig, 1922], toutes les congruences en question forment des systèmes-R et constituent un quadruple conjugué. Les quatre surfaces focales correspondantes sont alors toutes des surfaces réglées appartenant à une même congruence linéaire, et l'on peut passer de l'une quelconque de ces surfaces réglées à une autre par une transformation-W. Dans l'espace réglé, on obtient, dans le sens de cinématique projective, un mouvement central qui permet de construire géométriquement les 'T-Figuren' qui dépendent d'une fonction à une variable. Si le complexe appartient aux types définis par 49 ou 50, les deux surfaces focales opposées du 'T-Figur' font parties, soit d'une même quadrique non dégénérée, soit d'un même cône du second degré, soit enfin (d'une façon duale) d'une même conique. *F. Şemin.*

Pogorelov, A. V. Continuous maps of bounded variation. Dokl. Akad. Nauk SSSR (N.S.) **111** (1956), 757–759. (Russian)

Let S and S' be surfaces in E^3 locally representable in the form $z=\varphi(x, y)$, $\varphi \in C'$. A continuous mapping f of S on S' is called of bounded variation if the sum of the measures $|f(F_k)|$ of the images $f(F_k)$ of any finite set $\{F_k\}$ of non-overlapping closed sets F_k on S is uniformly bounded. For any open set G on S the absolute variation $V_f^0(G)=\sup \sum |f(F_k)||$ for all $\{F_k\} \subset G$, and for an arbitrary set M on F we put $V_f^0(M)=\inf_{G \supset M} V_f^0(G)$. Then $V_f^0(M)$ is totally additive on the Borel sets of S. If $u_M(x)$ denotes for $x \in S'$ the number of pre-images of x in M, then $\int_{S'} u_M(x)dx=V_f^0(M)$. A point x of S is regular relative to f if it has a neighborhood U with $f(x) \neq f(y)$ for $y \in U-x$. The total variation of f vanishes on the set of non-regular points. Assume S and S' to be homeomorphic to the plane and oriented. The index of f with respect to a regular point is defined. The positive and negative variations $V_f^+(M)$ and $V_f^-(M)$ of f on M are the variations of f on the points of M with positive and negative index. The variation of f on M is defined by

$$V_f(M)=V_f^+(M)-V_f^-(M).$$

Let the domain G on S be homeomorphic to a circular disk and bounded by the curve C with $V_f^0(C)=0$. Then $V_f(G)=\int_{S'-f(C)} q_{f(C)}(x)dx$, where $q_{f(C)}(x)$ is the index or winding number of x with respect to the curve $f(C)$.

If S' is the unit sphere and f is the mapping of S on S' by parallel normals, i.e., the spherical mapping, then V_f^0, V_f, V_f^+, V_f^- become respectively the integral absolute, the integral, the positive part, the negative part of the integral extrinsic curvature, and S is a so-called surface with bounded curvature. The points with index 1, 0, −1 are the elliptic, parabolic, hyperbolic points. If the integral absolute curvature vanishes, then S is locally isometric to the plane. (Cf. the review below.)

H. Busemann (Los Angeles, Calif.).

Pogorelov, A. V. Extensions of the theorem of Gauss on spherical representation to the case of surfaces of bounded extrinsic curvature. Dokl. Akad. Nauk. SSSR (N.S.) **111** (1956), 945–947. (Russian)

A surface of bounded extrinsic curvature (b.e.c.) is essentially a surface in E^3, locally representable in the form $z=f(x, y)$ with $f \in C'$, whose spherical image (multiplicities counted) has bounded measure. If $\sigma^+(H)$, $\sigma^-(H)$ denote the positive and negative parts of the extrinsic, $\omega^+(H)$ and $\omega^-(H)$ those of the intrinsic, curvature, then for any Borel set H on a surface of b.e.c. $\sigma^+(H)=\omega^+(H)$, $\sigma^-(H) \leqq \omega^-(H)$. The equality $\sigma^-(H)=\omega^-(H)$ holds for closed surfaces and for Borel sets in sufficiently small neighborhoods of regular points. If ϕ has b.e.c., $\omega^-(\phi)=0$, and ϕ has plane boundary, then ϕ is convex. If ϕ has b.e.c., and is locally isometric to a plane and complete, then it is a cylinder. *H. Busemann.*

★ **Whitney, Hassler. Geometric integration theory.** Princeton University Press, Princeton, N. J., 1957. xv+387 pp. $8.50.

Integration, in the sense of an extended theory of differential forms, is developed here with an approach dual to that of the theory of currents (de Rham, L. Schwartz). An r-cochain is a linear function of polyhedral r-chains in Euclidean E^n. Integration enters by assuming continuity conditions on the cochain (so that then one can, by approximation, use the cochain to integrate over more general domains, such as curved surfaces); one requires the cochain to be bounded with respect to one of two norms in the linear space of polyhedral r-chains. The flat norm is defined by $|A|^\flat=\inf\{|A-\partial D|+|D|:D$ an $(r+1)$-chain$\}$, where $|\ |$ is the "mass" of a chain derived in the natural way from the r-volume. The sharp norm $|A|^\sharp$ takes into account the possibility of translating pieces of A in various directions; $|A|^\sharp \leqq |A|^\flat$ holds. By completion one gets the spaces of flat, resp. sharp, chains; their duals are the flat, resp. sharp, cochains. As an example, a flat 0-cochain is identical with a "real sharp" function, i.e., a real-valued bounded function with a uniform Lipschitz condition. These cochains are now identified with certain types of differential forms. The minor result (Thm. 10A) makes sharp cochains correspond to sharp forms; these are bounded and satisfy a Lipschitz condition with respect to a certain norm in the space of forms, called the comass. The major result (Thm. 7C due to J. H. Wolfe) identifies flat cochains with flat forms; these are forms that are measurable, and bounded in a certain sense. For $r=n$ this is classical theorem on differentiation of set functions, and identifies flat n-forms with bounded measurable functions. For $r=1$ it leads to Rademacher's theorem on differentiability of Lipschitz functions. The importance of flat cochains derives from the fact that the exterior derivative can be defined (as dual of the ordinary boundary in the space of chains); by the identification this defines the exterior derivative of a flat form. Similarly the exterior product exists for flat cochains; and for a complex $K \subset E^n$ there is an isomorphism between the flat cohomology ring and the usual algebraic cohomology ring. The "reasonable" maps of this theory are the Lipschitz maps; they preserve flat cochains, and have the usual properties. The theory is developed in many directions, e.g., the connection between sharp r-chains and measurable functions whose values are r-vectors. In addition there is given a development of Grassmann algebra, a treatment of differential forms in the classical sense including things like the implicit function theorem and differential forms in a smooth manifold, integration of continuous forms over simplexes by a straightforward Riemann-type process, a proof of Stokes' theorem for reasonable $(n-1)$-forms over domains with reasonable boundary in E^n that seems very well adapted to the usual applications. Finally, there is a chapter with the theorem that smooth (separable) manifolds can be imbedded in

Euclidean space, and utilizing this fact, a quite simple and straightforward proof for triangulability of smooth manifolds. An elementary proof of de Rham's theorem and some remarks on the Hopf invariant conclude this chapter. The introductory chapter provides the motivation for the general theory and treats the 3-dimensional case in detail.

Table of contents: Introduction; Grassmann algebra; Differential forms; Riemann integration theory; Smooth manifolds; Abstract integration theory; Some relations between chains and functions; General properties of chains and cochains; Chains and cochains in open sets; Flat cochains and differential forms; Lipschitz mappings; Chains and additive set functions; Appendices: Vector and linear spaces; Geometric and topological preliminaries; Analytical preliminaries. *H. Samelson.*

Thom, R. Les singularités des applications différentiables. Ann. Inst. Fourier, Grenoble **6** (1955–1956), 43–87.

Let V^n be an n-dimensional differentiable manifold and let f be a differentiable real-valued function defined on V^n. The locus of critical points of f can be a rather complicated subset of V^n, and the behavior of f in the neighborhood of a critical point may be quite weird. However, the author proves the following theorem about this situation: The function f can be approximated arbitrarily closely by a differentiable function g such that g has only isolated critical points at each of which the matrix of second partial derivatives is non-singular. If desired, one may also demand that the partial derivatives of g up to a given order approximate the partial derivatives of f. [The author points out that this theorem is essentially due to M. Morse, "The calculus of variations in the large", Amer. Math. Soc. Colloq. Publ., v. **18**, New York, 1934, Th. 8.1, p. 178. Recently H. Whitney has also given a proof; see Ann. of Math. (2) **62** (1955), 374–410; MR **17**, 518].

The main objective of the present paper is to establish analogous theorems about the singularities of a differentiable map f of the manifold V^n into a p-dimensional differentiable manifold M^p. For any such map, let S_r, $r\geqq 0$, denote the set of all points $x\in V^n$ such that the matrix of partial derivatives of f has rank $p-r$ at the point x. With this notation, S_0 denotes (in case $n\geqq p$) the set of regular points of f, and the closure of S_1 is the set of all singular points of f. In general, the singular sets $S_1, S_2, S_3, \cdots$, can present a quite complicated structure. However, the author defines a notion of a "generic" map $f\colon V^n\to M^p$ (the definition is too complicated to reproduce here) and the structure of the singular sets S_r of a generic map is much simpler. He proves that any differentiable map can be approximated arbitrarily closely by a generic map, with approximation also on the partial derivatives up to a given order if desired. For a generic map $f\colon V^n\to M^p$, the singular set S_r ($r>0$) is a differentiable submanifold of dimension $n-r(n-p+r)$ if it is non-empty. Of course it need not be a closed submanifold; S_{r+1} is contained in the closure of S_r. In case $r(n-p+r)>n$, S_r is empty for all generic maps $f\colon V^n\to M^p$. In the proof of these results, the author uses methods he introduced in a previous paper [Comment. Math. Helv. **28** (1954), 17–86; MR **15**, 890].

Next, the author considers the behavior of the given map $f\colon V^n\to M^p$ when restricted to one of the critical sets S_r. The set of points $x\in S_r$ such that f restricted to S_r has its rank reduced q units from the maximum possible at x is denoted by $S_q(S_r)$. The points $S_0(S_r)$ are the regular points of f restricted to S_r; such points are called "ordinary critical points of f". Any point of $S_q(S_r)$ with $q>0$ is called an "exceptional critical point of f". For a generic map, the exceptional critical sets $S_q(S_r)$ ($q=1, 2, 3, \cdots$) are differentiable submanifolds of S_r (or else empty).

One can then consider the map f restricted to an exceptional critical set $S_q(S_r)$; the points $x\in S_q(S_r)$ where this map does not have maximal rank are called "surexceptional critical points of f". Thus one defines subsets $S_t(S_q(S_r))$ ($t=0, 1, 2, \cdots$) and so on. The definition of a generic map is so framed that each surexceptional critical set $S_t(S_q(S_r))$ is a submanifold of $S_q(S_r)$ in case f is generic.

The author next considers the various critical sets in more detail in the case of a generic map $f\colon V^n\to R^p$, where R^p denotes p-dimensional Euclidean space, and $p\leqq 4$. In case $p=2$, he obtains the following result: For a generic map $f\colon V^n\to R^2$, the only non-void critical set is the set S_1, which is 1-dimensional (a non-singular curve). The map f restricted to S_1 has isolated critical points, which constitute the set $S_1(S_1)$. The exceptional critical points $S_1(S_1)$ give rise to cusps on the image of the curve S_1 in R^2. [This result is also contained in the paper by Whitney mentioned above in case $V^n=R^2$.] In an analogous manner, any generic map $f\colon V^n\to R^3$ has a critical set S_1 which is a non-singular surface, an exceptional critical set $S_1(S_1)$ which is a curve on S_1, and surexceptional critical points $S_1(S_1(S_1))$. The image of $S_1(S_1)$ is a curve of cusps on the surface which is the image of S_1, and the image of $S_1(S_1(S_1))$ is the set of cusps on this curve. The author also considers the introduction of certain canonical coordinate systems in the neighborhood of these various singularities such that the function f takes a particularly simple form with respect to these coordinates.

Lack of space prevents the reviewer from indicating the many properties of the singularities of a generic map discussed by the author. The following three examples, will give the reader some idea of the questions considered, however: (1) May any one of the above-mentioned singular sets actually be attained (i.e., is it non-empty?) for an appropriately chosen generic map $f\colon V^n\to M^p$? (2) Are these various singularities stable under all sufficiently small deformations of a given generic map f? (3) For a generic map $f\colon V^n\to M^p$, the closure of the critical set S_r is a pseudomanifold, and hence may be considered as a cycle mod 2. What is its homology class mod 2? What does it depend on? How is it related to the various characteristic classes of V^n, such as Stiefel-Whitney classes and Pontrjagin classes? Of course one can ask analogous questions for the exceptional critical sets $S_q(S_r)$, and so on.

This paper is written in a semi-expository style. Not all the definitions are stated precisely; the proofs of many theorems are only sketched; there are many conjectures listed throughout the paper; etc. However, it is a pioneering paper in what is probably a vast new field of research, and in such a situation this style of writing may well be best. It is to be hoped that at some future date the author will be able to write a more precise account of this subject. *W. S. Massey* (Providence, R.I.).

Gugenheim, V. K. A. M.; and Spencer, D. C. Chain homotopy and the de Rham theory. Proc. Amer. Math. Soc. **7** (1956), 144–152.

The paper begins with a concisely presented invariant approach to the study of differential forms on manifolds, including manifolds with almost product structure. The

slant operation ϕ/c assigns to an $(r+s)$-form ϕ on a product manifold $U\times V$ and to a singular r-chain c in U an s-form on V. Using this operation a chain homotopy operator is introduced and used to show that differentiably homotopic maps induce chain homotopic homomorphisms (a generalization is given for almost product structure), which establishes the homotopy axiom for differential forms and thus proves the Poincaré Lemma. A similar proof was given by G. de Rham [G. de Rham and K. Kodaira, Harmonic integrals, Inst. Advanced Study, 1950, Princeton; MR 12, 279].

Manifolds with complex or almost complex structure are then studied by the introduction of potential theoretic methods. In special cases results similar to those for real manifolds are obtained, but in general the chain homotopy does not extend, as is shown by an example due to Kodaira for which $\bar{\partial}$ cohomology does not satisfy the homotopy axiom. *M. P. Gaffney* (New York, N.Y.).

Lingenberg, Walter. Zur Differentialgeometrie der Flächen, die eine eingliedrige projektive Gruppe in sich gestatten und über Allgemeine Projektivrotationsflächen. Math. Z. **66** (1957), 409–446.

In der vorliegenden Arbeit werden im drei-dimensionalen Raum die sog. Π-Flächen, d.h., Flächen, die eine eingliedrige stetige Gruppe von projektiven Transformationen in sich gestatten und keine Regelflächen sind, betrachtet. Diese bereits insb. von Lie behandelten Flächen kommen in der neueren Differentialgeometrie häufig vor, indem sie in verschiedenen Flächenklassen mit vorgegebenen Eigenschaften eine ausgezeichnete Stellung einnehmen. Die vorliegende Arbeit ist besonders vom methodischen Standpunkt wertvoll, da in ihr mehrere durchaus nicht triviale Integrationsprozesse restlos durchgeführt werden. Die theoretischen Untersuchungen werden auf Grund des von Bol für Asymptotenlinienparameter aufgestellten Kalküls der halbinvarianten Differentiation durchgeführt, während für die Integrationsprozesse ein zu den Asymptotenlinien konjugiertes Parameternetz, dessen eine Schar von Parameterlinien aus Bahnkurven der zugehörigen eingliedrigen Gruppe besteht, verwendet wird. Die in der Arbeit behandelten Fragen stehen insb. mit den Resultaten von Blaschke, Čech, B. Segre, B. Su und dem Ref. im Zusammenhang. Durch Integrationsprozesse werden aus dem vollständigen System von Halbinvarianten β, γ, l, m insb. Parameterdarstellungen für allgemeine sowie für gewisse im Jahre 1929 vom Ref. mit Hilfe der Cartanschen Methoden gefundene projektiv abwickelbare Π-Flächen abgeleitet. Von den übrigen Resultaten wollen wir insb. anführen, dass die isotherm-asymptotischen Π-Flächen sämtlich projektiv abwickelbar sind, und die folgende Charakterisierung der Projektivrotationsflächen: Wenn bei einer Fläche sämtliche kanonischen Normalen erster und zweiter Art durch je eine feste Gerade hindurchgehen, so ist die Fläche eine Projektivrotationsfläche und umgekehrt. Von Regelflächen wird dabei abgesehen.

O. Borůvka (Brünn).

Nožička, František. Ein Beitrag zur Affingeometrie der Fläche. Časopis Pěst. Mat. **81** (1956), 137–156. (Czech. Russian and German summaries)

Mayer, O. Surfaces applicables en géométrie centro-affine hyperbolique. An. Şti. Univ. "Al. I. Cuza" Iaşi. Secţ. I. (N.S.) **1** (1955), 137–164. (Romanian. Russian and French summaries)

From the French summary: "A centro-affine isometry is a correspondence between two surfaces which preserves the first fundamental form F in the theory of surfaces of the centro-affine space. [$F=(x_u\ x_v\ d^2x)/(x\ x_u\ x_v)$ was introduced by the author in Ann. Sci. Univ. Jassy **21** (1935), 1–77.] This is equivalent to the conservation of the centro-affine arc length of circumscribed strips and the direction of their concavity with respect to O (the center of the space).

"Each regular analytic surface, which is not a cylinder or a cone of vertex O, is susceptible to isometric deformations which depend upon 3 arbitrary functions of an argument.

"The surfaces which admit a continuous group ∞^1 of isometries onto themselves are applicable on surfaces of 'revolution'. The surfaces of revolution having constant curvature K are determined. The surfaces $K=-1$ are applicable on non-degenerate quadrics of center O and the surfaces $K=0$ on paraboloids passing through O. Finally, the applicability of ruled surfaces is studied."

R. Blum (Saskatoon, Sask.).

Strubecker, Karl. Über die Flächen $rt-s^2=K=$ konst. und ihren Zusammenhang mit den Flächen $Kr+t=0$. Abh. Math. Sem. Univ. Hamburg **21** (1957), 99–103.

Die Paraboloide (*) $2z=a_{11}x^2+2a_{12}xy+a_{22}y^2$, $D=a_{11}a_{22}-a_{12}^2\neq 0$, werden bekanntlich in der vom Verfasser eingehendst entwickelten Theorie des isotropen Raumes als Kugeln aufgefasst und jeder auf einem solchen Paraboloid liegender Streifen als isotroper sphärischer Streifen. Sind dann $K=rt-s^2$ und $K^*=r^*t^*-s^{*2}$ die isotropen Relativkrümmungen zweier bezüglich (*) polarer Flächen Φ und Φ^*, so gilt $K^*=D^2K^{-1}$. Daher ist die Fläche $z=z(x,y)$ zur isotropen Kugel (*) selbstpolar, wenn sie einen auf (*) liegenden sphärischen Streifen enthält. — Das Gegenstück der Lieschen Geraden-Kugeltransformation ist im isotropen Raum durch die Eulersche Berührungstransformation gegeben. Die auf Flächenelemente zweiter Ordnung erweiterte Eulersche Berührungstransformation

$$(x, y, z, p, q, r, s, t)\to(\bar{x}, \bar{y}, \bar{z}, \bar{p}, \bar{q}, \bar{r}, \bar{s}, \bar{t})$$

vermittelt einen Zusammenhang zwischen den Flächen der festen Relativkrümmung $rt-s^2=K$ und den Integralflächen der Gleichung $K\bar{r}+\bar{t}=0$. Insbesondere werden die Integralflächen von $rt-s^2=\pm 1$ in die von $\bar{r}\pm\bar{t}=0$ transformiert, d.h., in Schiebflächen mit zwei Scharen konjugiert komplexer bzw. reell ebener Schiebkurven. Ferner behält auch im isotropen Raum der bekannte Liesche Vertauschungssatz für Krümmungs- und Asymptotenlinien seine Gültigkeit: die Vertauschung wird durch das isotrope Gegenstück der Geraden-Kugeltransformation bewirkt. Schließlich wird noch gezeigt: enthält eine isotrope Minimalfläche ($r\pm t=0$) eine (reelle) nicht isotrope Gerade, so ist sie (im Sinne der isotropen Kinematik) bezüglich dieser Geraden symmetrisch. Im Rahmen der affinen Differentialgeometrie sind die Integralflächen der partiellen Differentialgleichung $rt-s^2=K=\text{const}\neq 0$ als „uneigentliche Affinsphären" von G. Darboux und W. Blaschke behandelt worden.

M. Pinl.

Popescu, I. Au sujet de l'arête de rebroussement de la développable rectifiante affine. Com. Acad. R. P. Romîne **5**, 471–478 (1955). (Romanian. Russian and French summaries)

See also: Topological Groups: Tits. **Geometries, Euclidean and Other:** Alexandrow. **Manifolds, Connections:** Pisareva; Okubo; Sasayama. **Algebraic Geometry:** Wallace.

Manifolds, Connections

Varga, O. Über Riemannsche Räume, die freie Beweglichkeit besitzen. Schr. Forschungsinst. Math. **1** (1957), 124–130.

The author furnishes a new and interesting demonstration of the result that a Riemann space V_n of n dimensions admitting a group of free motions is a space of constant curvature. The procedure is as follows. Associated with every vector and its image under the group is the invariant metric quadratic form. Further, every pair of vectors determines a bivector and hence a vector in a tangent space of $\frac{1}{2}n(n-1)$ dimensions. If g_{ij} is the metric tensor of V_n, then the tensor $g_{sk}g_{ml}-g_{sl}g_{mk}$ determines a metric tensor, G_{JP}, in a space of $\frac{1}{2}n(n-1)$ dimensions and a unique metric quadratic form which is invariant under the group of motions. The remainder of the proof depends upon showing that the curvature tensor R_{smkl} of V_n determines a quadratic form which is invariant under the group of motions. This is done by use of the Gauss equation for a V_{n-1} in the V_n. Finally, the author considers a V_n with the property that at each point in every $(n-1)$-direction a hypersurface exists with Euclidean metric. By considering an orthogonal n-tuple consisting of $n-1$ orthogonal vectors in the $(n-1)$-direction and a vector orthogonal to this direction, and studying the motions of this n-tuple from one point to another, the author shows that V_n is a space of constant negative curvature. *N. Coburn* (Ann Arbor, Mich.).

Hiramatu, Hitosi. On Riemannian spaces admitting groups of conformal transformations. J. Math. Soc. Japan **9** (1957), 114–130.

R_N is an N-dimensional Riemannian space with a positive definite metric. The author shows that in R_N for $N\geqq 3$, $N\neq 4$, there exists no group of conformal transformations of order r with $\frac{1}{2}N(N+1)+2<r<\frac{1}{2}(N+1)(N+2)$, and that, except for a finite number of N's, if R_N for $N\geqq 3$ admits a group of conformal transformations of order r such that $\frac{1}{2}(N-1)(N-2)+2<r<\frac{1}{2}N(N-1)+1$, then the R_N is conformally flat. The arguments used in the proof are in the main based on local properties of R_N. *A. H. Taub* (Urbana, Ill.).

Kručkovič, G. I. On motions in Riemann spaces V_4. Mat. Sb. N.S. **41(83)** (1957), 195–220. (Russian)

The problem of classification of Riemann spaces V_4 with respect to the groups of motions was initiated by G. Fubini [Ann. Mat. Pura Appl. (3) **8** (1903), 39–81; **9** (1904), 33–90]. Fubini's method was applicable to the case when the surface of transitivity of an intransitive group in V_4 is not isotropic. The present author completes the solution of the problem, investigating the Riemannian V_4's which admit an intransitive group of motions with isotropic surfaces of transitivity. The author solves this problem completely and in each separate case obtains the generators of the group and the fundamental quadratic form of the space in the appropriately chosen system of coordinates. *T. P. Anđelić* (Belgrade).

Kručkovič, G. I. On uniqueness of decomposition of a reducible Riemannian space. Dokl. Akad. Nauk SSSR (N.S.) **108** (1956), 583–586. (Russian)

Walker, A. G. Connexions for parallel distributions in the large. Quart. J. Math., Oxford Ser. (2) **6** (1955), 301–308.

Der Autor hat p-Richtungsfelder in V_n, die in sich parallel sind, schon 1949 [dasselbe J. **20** (1949), 135–145; MR **11**, 460] behandelt. Hier wird gezeigt, daß es zu jedem System solcher Felder der Klasse C^∞ in einer X_n der Klasse C^∞ lineare Übertragungen gibt, in bezug auf welche diese Felder alle in sich parallel sind, und daß es unter diesen im integrablen Falle (d.h., wenn jedes p-Feld X_p bildend ist) auch symmetrische Übertragungen gibt.

J. A. Schouten (Zbl. **66** (1957), 402).

Strubecker, Karl. Differentialgeometrie isotroper Mannigfaltigkeiten. Schr. Forschungsinst. Math. **1** (1957), 143–155.

This is a report on analytical V_m, $\bar{x}=\bar{x}(u^1, u^2, \cdots, u^m)$ of complex euclidean space $R_n(x^1, x^2, \cdots, x^n)$ with a $ds^2=g_{\alpha\beta}du^\alpha du^\beta$ of which the tensor $g_{\alpha\beta}$ has rank $r<m$. Such V_m are called $(m-r)$-times isotropic, or isotropic of rank v; for $r=0$ the V_m are totally isotropic or ametric. In this last case there exists, for $m=2$, a quasimetric based on a differential form of order 4. Of particular interest is the differential geometry of linear isotropic spaces J_m of different rank; the case $m=3$ is discussed in detail. Reference is made to work by Beck, Berwald, Lense, Pinl, Scheffers, Study, C. L. E. Moore, the author, and some others. A complementary bibliography can be found on p. 227 of Schouten's "Ricci-calculus" [2nd ed., Springer, Berlin, 1954; MR **16**, 521]. *D. J. Struik.*

Golab, S. Zur Theorie der Übertragungen. Schr. Forschungsinst. Math. **1** (1957), 162–177.

A condensed report on the theory of displacements and connections in manifolds of n dimensions since its beginnings in 1917. Special attention is paid to geometrical objects, the geometry of paths, extensors, and Finsler geometries with their generalizations, notably those of the Kawaguchi school. *D. J. Struik* (Cambridge, Mass.).

Pisareva, N. M. Spaces of Weyl enclosing a projective system of paths. Mat. Sb. N.S. **41(83)** (1957), 231–238. (Russian)

In connection with a previous paper by the author [Mat. Sb. N.S. **36(78)** (1955), 169–200; MR **16**, 748] dealing with a Weyl space characterized by the property that its equations of geodesic admit quadratic fractional first integrals (this space contains in itself two mutually orthogonal systems of totally geodesic surfaces), the present paper is concerned with Weyl spaces enclosing a projective system of paths, and with an explanation of the relation between these spaces and reduced Weyl spaces. It is shown that both classes of spaces have a common subclass of spaces characterized by the property that a space of this subclass has two mutually orthogonal systems of totally geodesic surfaces, where one of these systems determines a geodesic field of m-dimensional directions in the space under consideration. Finally, the fundamental metric form and the supplementary vector of the Weyl space are shown to enclose a projective system of paths. *A. Kawaguchi.*

Scheibe, Erhard. Über das Weylsche Raumproblem. J. Reine Angew. Math. **197** (1957), 162–207.

The author first defines and discusses in detail analytic fibre spaces and group fields, connections and group metrics in fibre spaces, and then proves the following Weyl's theorem. Postulate (W): Every linear group metric of a class Q of closed subgroups of the full linear group L_n possesses just one compatible symmetric affine connection. Theorem: Q satisfies the Postulate (W) if there is a group $K \in Q$ such that, for $n \geqq 3$, the Lie algebra of K is the totality of matrices of the form $\rho = \gamma^{-1}\alpha$, ${}^t\gamma = \gamma$, $\det(\gamma) \neq 0$, ${}^t\alpha + \alpha = 0$, with fixed γ and arbitrary α, and, for $n=2$, the algebra is one of the following four matrices:

$$\begin{pmatrix} a & -1 \\ 1 & a \end{pmatrix}; \quad \begin{pmatrix} a & 1 \\ 1 & a \end{pmatrix}, \ a^2 - 1 \neq 0;$$

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \ a \neq 0; \quad \begin{pmatrix} a & 0 \\ 1 & a \end{pmatrix}, \ a \neq 0.$$

The author also expresses the theorem in Weyl's original form [H. Weyl, Mathematische Analyse des Raumproblems, Springer, Berlin, 1923, pp. 51–61, 86–112] by using an equivalent of the postulate (W), and gives another proof of the theorem in the latter form by following E. Cartan's idea [C. R. Acad. Sci. Paris **175** (1922), 82–85; J. Math. Pures Appl. (9) **2** (1923), 167–192].

C. C. Hsiung (Bethlehem, Pa.).

Ślebodziński, W. Sur l'équivalence des formes différentielles extérieures du second degré. Rozprawy Mat. **11** (1956), 34 pp.

The equivalence problem for quadratic exterior differential forms (or for skew symmetric, second order covariant tensors) is the following: given $\Omega = \sum a_{\alpha\beta} dx^\alpha \wedge dx^\beta$ and $\bar{\Omega} = \sum \bar{a}_{\alpha\beta} d\bar{x}^\alpha \wedge d\bar{x}^\beta$ defined in neighborhoods of a point in $(x^1, \cdots, x^m)$, $(\bar{x}^1, \cdots, \bar{x}^m)$-space respectively, $m = 2n$, find necessary and sufficient conditions that there exist a change of coordinates $\bar{x}^\alpha = f^\alpha(x^1, \cdots, x^m)$ which carries $\bar{\Omega}$ into Ω. It is assumed throughout that the forms are of highest rank, i.e. $|a_{\alpha\beta}| \neq 0$ or equivalently $\Omega^n \neq 0$; similarly $|\bar{a}_{\alpha\beta}| \neq 0$. For this reason Ω is somewhat like a metric tensor; in particular, it affords a natural isomorphism between the tangent space and its dual at each point, that is, it can be used to raise or lower indices. Although the similar problem for a covariant, second order, positive-definite, symmetric tensor was solved by Christoffel [J. Reine Angew. Math. **70** (1869), 46–70] and Lipschitz [ibid. **70** (1869), 71–102], the problem for skew symmetric tensors has been solved only in special cases. The author gives in fact two solutions of the general case, although the first is only sketched, except where some special assumptions are made, in which case a detailed treatment is given.

The first method considers first the equivalence problem for a pair of quadratic forms Ω and Φ. In this case the functions A_i given by $A_i dx^1 \wedge \cdots \wedge dx^m = \Omega^{n-i} \wedge \Phi^i$ are tensor densities and $A_1/A_0, \cdots, A_n/A_0$ are invariants. Assuming these are independent functions the author determines all of the differential invariants of the pair Ω, Φ. This is applied to the original problem as follows. Associated with Ω there is a canonically determined quadratic form Ψ. In fact, let $d\Omega = S_{\alpha\beta\gamma} dx^\alpha \wedge dx^\beta \wedge dx^\gamma$ and $\sigma = \sum a^{\beta\gamma} S_{\alpha\beta\gamma} dx^\alpha$, $\|a^{\beta\gamma}\|$ being the inverse of $a_{\alpha\beta}$, coefficients of Ω. Then $\Psi = d\sigma$. Applying results obtained above to the case $\Phi = \Psi$ gives the first solution, i.e., Ω, $\bar{\Omega}$ are equivalent if and only if the systems Ω, Ψ and $\bar{\Omega}$, $\bar{\Psi}$ have the same invariants. As a corollary the author determines the (infinite) Lie group of transformations of a domain $(x^1, \cdots, x^m)$ which preserve such a form Ω.

The second treatment of the problem is analogous to that of E. Cartan for the positive definite, symmetric case [see § 6, S. S. Chern, Bull. Amer. Math. Soc. **52** (1946), 1–30; MR 9, 101]. Remark first that the structure group of the tangent bundle of a space with a quadratic exterior form Ω of maximum rank may be reduced to the real symplectic group. In the resulting principal fibre bundle there exist many affine connections; however, the author proves that the connection is uniquely determined if it is required that the curvature be zero and the torsion tensor (written with covariant indices) satisfy $S_{\alpha\beta\gamma} = S_{\beta\gamma\alpha} = S_{\gamma\alpha\beta}$. The proof of this is not trivial and depends on showing that the exterior differential system given by the structure equations is, under these assumptions, in involution and has a unique solution. This being done, the forms ω^α, where $\Omega = \omega^1 \wedge \omega^{n+1} + \cdots + \omega^n \wedge \omega^{2n}$, and $\omega_\alpha{}^\beta$ the torsion forms, are intrinsically defined in the bundle, and then Ω, $\bar{\Omega}$ are equivalent if and only if $\omega^\alpha = \bar{\omega}^\alpha$, $\omega_\alpha{}^\beta = \bar{\omega}_\alpha{}^\beta$.

W. M. Boothby (Evanston, Ill.).

Okubo, Tanjiro. On the order of the groups of affine collineations in the generalized spaces of paths. I. Tensor (N.S.) **6** (1956), 141–158.

Okubo, Tanjiro. On the order of the groups of affine collineations in the generalized spaces of paths. II, III. Tensor (N.S.) **7** (1957), 1–17, 18–33.

These three papers constitute three parts of one long paper with consecutively numbered chapters and theorems. The objects of study are generalized spaces of paths, i.e., manifolds in which are defined locally equations of the form $d\dot{x}^i/dt + \Gamma^i_{jk}(x, \dot{x})\dot{x}^j\dot{x}^k = 0$, where $\dot{x}^i = dx^i/dt$, the Γ^i_{jk} are symmetric in j, k, homogeneous of degree zero in $\dot{x}^i$, and transform under change of coordinates by the usual law for affine connections. Their solutions are the paths of the space and a transformation of the space mapping paths onto paths is called an affine collineation. The largest group of such transformations, G_r, is finite of order r, and in these papers the author studies a problem stated by M. S. Knebelman [Amer. J. Math. **51** (1929), 527–564], to wit: classify these spaces according to the order r of G_r. The maximum order is $r = n^2 + n$ (where n is the dimension of the manifold). The author shows that (1) $n^2 + n \geqq r > n^2$ is impossible except for the ordinary affine space A_n, (2) if $n \geqq 7$, then $n^2 \geqq r > n^2 - n + 1$ is also impossible (except for A_n), and (3) there exist examples where $r = n^2 - n + 1$, $n \geqq 7$. He also gives some special results for $n < 7$. The method consists of a lengthy and detailed examination of the conditions for complete integrability of the partial differential equations $\pounds_\xi \Gamma^i_{jk} = 0$, $\xi^i{}_{,j} = \partial\xi^i/\partial x^j + \Gamma^i_{hj}\xi^h$, where $\bar{x}^i = x^i + \xi^i(x)\delta u$ is an infinitesimal transformation relative to the contravariant vector field $\xi^i(x)$, $\pounds$ denotes the Lie derivative, and these differential equations give the condition that it be an infinitesimal affine collineation. For, if the equations of this system, including all equations which can be adjoined by successive differentiations, depend linearly on $n^2 + n - r$ equations among them, these being homogeneous in ξ^i, $\xi^i{}_{,j}$, then the group G is finite of order r [Knebelman, op.cit.].

W. M. Boothby (Evanston, Ill.).

Kobayashi, Shoshichi. Canonical connections and Pontrjagin classes. Nagoya Math. J. **11** (1957), 93–109.

Let M be a differentiable manifold of dimension n and P the bundle of frames (with group $\mathrm{GL}(n, R)$) over M.

A group G of differentiable transformations acting on M is called almost effective if the subgroup N consisting of all elements in G which leave M pointwise fixed is discrete. If G is almost effective and there exists an affine connection on M invariant under G, every connection in the bundle G can be considered as an affine connection on M. These general results are applied to the Grassmann manifold $M_{n,k}=O(n+k)/O(n)\times O(k)$ acted on by the orthogonal group $O(n+k)$ in $n+k$ variables. In particular, the canonical connection in the bundle $O(n+k)$ is defined, which corresponds to the invariant Riemannian connection on $M_{n,k}$. These connections and their relationship are given explicitly. The author next determines the holonomy groups, the curvature forms, and the characteristic classes of these connections. Of importance are three kinds of characteristic classes corresponding to three kinds of symmetric functions, among which are the Pontrjagin characteristic classes. From these algebraic expressions of the Pontrjagin characteristic classes in terms of curvature forms the author derives a proof of the duality theorem between the tangential and normal Pontrjagin characteristic classes (with real coefficients) of an imbedded manifold in Euclidean space. *S. Chern* (Chicago, Ill.).

Sasayama, Hiroyoshi. On the intrinsic derivative of generalized order. Math. Mag. **30** (1957), 135–143.

In this paper the author gives a generalization of the notion of derivative introduced by E. L. Post [Trans. Amer. Math. Soc. **32** (1930), 723–781]. He first shows that the excovariant and crossed extensors of integral infinite ranges are derived from tensors by the operation of differentiation as introduced by Post. He then constructs the derivatives of covariant tensors in a non-holonomic space with an affine connection. Some properties of these derivatives are then given. The author uses the theory of extensors as developed by Craig [Amer. J. Math. **59** (1937), 764–774; see also MR **5**, 77; **8**, 491; **9**, 159; **11**, 543; **14**, 1016]. *E. T. Davies* (Southampton).

Laptev, B. L. The Lie derivation in the space of supporting elements. Trudy Sem. Vektor. Tenzor. Anal. **10** (1956), 227–248. (Russian)

Ōtsuki, Tominosuke. On geodesic coordinates in Finsler spaces. Math. J. Okayama Univ. **6** (1957), 135–145.

Let M^n be an n-dimensional Finsler space ($n>4$). The author computes a tensor whose vanishing identically is a necessary and sufficient condition for the geodesic coordinate system along each geodesic arc to be of class C^2. *L. Auslander* (Bloomington, Ind.).

★ **Chern, Shiing-shen. On a generalization of Kähler geometry.** Algebraic geometry and topology. A symposium in honor of S. Lefschetz, pp. 103–121. Princeton University Press, Princeton, N. J., 1957. $7.50.

Es sei M eine n-dimensionale orientierte kompakte differenzierbare Mannigfaltigkeit mit Riemannscher Metrik. Durch die Metrik ist die Strukturgruppe des Tangentialbündels von M auf die orthogonale Gruppe $SO(n)$ reduziert. G sei eine abgeschlossene Untergruppe von $SO(n)$. Die Strukturgruppe des Tangentialbündels ist dann und nur dann auf G reduzierbar, wenn M einen affinen Zusammenhang besitzt, dessen Holonomiegruppe in G enthalten ist. Man nehme an, dass ein solcher G-Zusammenhang vorgegebenen ist, man hat dann ein G-Prinzipalbündel ξ über M, das durch Erweiterung der Strukturgruppe in das tangentielle $SO(n)$-Bündel übergeht. $SO(n)$ operiert in kanonischer Weise auf $V=R^n$ und damit auf dem Raum $\Lambda^p(V^*)$ der kovarianten p-Vektoren. Als Untergruppe von $SO(n)$ operiert auch G auf diesem Raum, der dann eine Zerlegung $\Lambda^p(V^*)=V_1+V_2+\cdots+V_r$ zulässt, wo die V_i zueinander orthogonale invariante Teilräume von $\Lambda^p(V^*)$ sind, in denen G durch irreduzible Darstellungen operiert. Vermöge dieser Darstellungen hat man zu ξ assoziierte Vektorraum-Bündel W_i mit V_i als Faser, und das Vektorraum-Bündel $T^{(p)}$ der kovarianten p-Vektoren von M ist die Whitneysche Summe der W_i. Jede Differentialform w auf M vom Grade p (d.h. jeder Schnitt von $T^{(p)}$) kann auf kanonische Weise dargestellt werden in der Form (*) $w=P_1w+P_2w+\cdots+P_rw$, wo P_iw ein Schnitt von W_i ist. Zu der Riemannschen Metrik von M gehört der Operator $\Delta=d\delta+\delta d$. In allgemeinen ist $\Delta P_i\neq P_i\Delta$; und für eine harmonische Form w werden die Summanden P_iw der Summe (*) im allgemeinen nicht harmonisch sein. Das Hauptresultat des Verf. besagt folgendes: Wenn M einen G-Zusammenhang zulässt, der einen verschwindenden Torsionstensor hat, dann ist P_i mit Δ vertauschbar und damit sind die Summanden der Summe (*) für eine harmonische Form wieder harmonisch.

Verf. wendet sein Resultat auf $n=2m$ und $G=U(m)$ an. Das G-Prinzipalbündel ξ liefert dann eine fast-komplexe Struktur von M. Wenn M eine kählersche Mannigfaltigkeit ist, dann hat M einen $U(m)$-Zusammenhang mit verschwindendem Torsionstensor, und der Zerlegungssatz des Verf. geht über in den klassischen Satz von Hodge über die Zerlegung einer harmonischen Form vom Typ (p, q) in harmonische Formen verschiedener Klasse. Man hat dazu die Darstellung von $U(m)$ im Vektorraum der kovarianten $(p+q)$-Vektoren vom Typ (p, q) in irreduzible Bestandteile zu zerlegen, was Verf. mit Hilfe der Charakter-Formel von H. Weyl durchführt. Verf. wendet sein Resultat insbesondere noch auf $G=SO(k)\times SO(n-k)$ an. Die Überlegungen des Verf. lassen weitere interessante Anwendungen zu. *F. Hirzebruch.*

See also: Functions of Complex Variables: Hua. **Geometries, Euclidean and Other:** Nottrot; Alexandrow.

Complex Manifolds

Dalla Volta, Vittorio. Varietà totalmente geodetiche nello spazio delle matrici simmetriche. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **18** (1955), 619–622.

Statement of results of the paper in Rend. Mat. e Appl. (5) **13** (1955), 294–334 [MR **18**, 934].

Griesel, Heinz. Überkonvergenz in der Funktionentheorie mehrerer Veränderlichen. Schr. Math. Inst. Univ. Münster no. **12** (1957), i+66 pp.

Die Überkonvergenz der Potenzreihen einer komplexen Veränderlichen ist während der Jahre 1920–1940 in mannigfache Publikationen untersucht worden. Eine Untersuchung der gleichen Erscheinung an Potenzreihen mehrerer Veränderlichen war jedoch bislang nur in einer Abhandlung von Lösch [Math. Z. **36** (1932), 202–262] durchgeführt worden. Der Verf. der vorl. Arbeit überträgt nun die Überkonvergenzsätze der klassichen Funktionentheorie auf Potenzreihen zweier Veränderlichen. Da bei zweifachen Potenzreihen Lücken verschiedenster Gestalt auftreten (z.B. Parallelstreifenlücken, Haken-

lücken, Diagonallücken) ergeben sich Sätze, die von der Art der Lücken abhängen (§ 2). In § 3 sind Umkehrungen von diesen Sätzen behandelt (z.B., dass die Überkonvergenz durch Lücken im Koeffizientenschema der Potenzreihen verursacht ist). Der § 4 zeigt, dass unter gewissen Voraussetzungen eine überkonvergente Reihe im ganzen Holomorphiegebiet der dargestellten Funktion gleichmässig konvergiert. Unter Verwendung dieses Resultates wird schliesslich der interessante Satz gezeigt, dass ein Gebiet GCC^2 (=zweidimensionaler komplexer Zahlenraum) genau dann ein Überkonvergenzgebiet ist, wenn es ein Rungesches Holomorphiegebiet ist. (Ein Überkonvergenzgebiet ist ein Gebiet GCC^2, das eine zusammenhängende Komponente des Konvergenzbereiches einer überkonvergenten Reihe ist.) *H. Grauert.*

Gunning, R. C. Multipliers on complex homogeneous spaces. Proc. Amer. Math. Soc. **8** (1957), 394–396.

On considère un group de Lie réel G d'automorphismes analytiques complexes opérant transitivement sur une variété analytique complexe D simplement connexe. Un multiplicateur $\mu(g; z)$ est une fonction C^∞ à valeurs complexes définie pour $(g; z) \in G \times D$ et vérifiant

$$\mu(g_1 g_2; z) = \mu(g_1; z)\mu(g_2; z);$$

on suppose G connexe; on désigne par $\mathcal{M}(G; D)$ le groupe abélien des multiplicateurs $\mu(g; z)$. On considère alors le groupe G^* recouvrement universel de G (on pose $g^* z = gz$ si $z \in D$ et $g^* \in G^*$ recouvre g), et le groupe $\mathcal{M}(G^*; D)$. Il contient le sous-groupe $\mathcal{J}(G^*; D)$ des multiplicateurs qui s'écrivent $\mu(g^*; z) = f(g^* z)[f(z)]^{-1}$, où $f(z)$ est holomorphe et ne s'annule pas sur D. On considère le groupe additif $\mathcal{L}(G^*; D)$ des formes de type (0, 1) sur D qui sont des $\bar\partial g(z, \bar z)$, $g(z, \bar z)$ étant une fonction C^∞ à valeurs complexes sur D (si D est une variété de Stein, il suffira de considérer les formes $\bar\partial$ fermées de type (0, 1) invariantes par G^*, la $\bar\partial$ cohomologie étant alors triviale). On désigne par K^* le sous-groupe de G^* qui laisse invariants les points de D et par $\mathrm{Hom}(K^*; C)$ le groupe des homomorphismes C^∞ de K^* dans le groupe additif des nombres complexes. Alors on établit l'isomorphisme

$$\mathcal{M}(G^*; D) \cong \mathcal{L}(G^*; D) \oplus \mathcal{J}(G^*; D) \oplus \mathrm{Hom}(K^*; C).$$

Application au cas où G est le groupe symplectique, D le domaine de Siegel $[I - Z\bar Z > 0]$; $\mathcal{L}(G^*; D)$ est un espace vectoriel à 1 dimension et la seule forme (1, 1) invariante fermée est la forme Ω déterminée par la métrique.

P. Lelong (Paris).

Grothendieck, A. Sur la classification des fibrés holomorphes sur la sphère de Riemann. Amer. J. Math. **79** (1957), 121–138.

Let X be the Riemann sphere. If G is a complex Lie group, denote by $B(G)$ the set of classes of holomorphic fiber bundles over X with structural group G. Denote by H a Cartan subgroup of G and by $W = N(H)/H$ ($N(H)$ being the normalizer of H) the Weyl group. We have a natural mapping $B(H) \to B(G)$; on the other hand, W operates on $B(H)$ and the image in $B(G)$ of an element $E \in B(H)$ is not changed if we replace E by $w(E)$, where $w \in W$. Thus we have a mapping j: $B(H)/W \to B(G)$, where $B(H)/W$ is the set of W-orbits in $B(H)$. The main theorem of the paper asserts that j is a bijection when G is reductive. The theorem is first established in the case where G is the full linear group; in that case, it says that a holomorphic fiber bundle over X whose fibers are vector spaces splits into a direct sum of bundles whose fibers have dimension 1 and whose degrees (degrees of the associated divisor classes on X) form a decreasing sequence. The theorem is then extended to the case of an arbitrary reductive group G by making use of the associated fiber bundle E whose fiber is the Lie algebra of G; this bundle is treated as a vector-space bundle and decomposed into one-dimensional vector bundles; the sub-bundle generated by the meromorphic sections whose degrees are ≤ 1 turns out to have a Lie algebra as its fiber, and the consideration of this Lie algebra is the key to the proof of the surjective character of the mapping j. To prove that j is injective, the author reduces the problem to the case where G is semi-simple; in that case, he applies the results relative to vector-space bundles to the bundles deduced from the given one by means of the linear representations of G.

Is is stated that the assumption that G is reductive is essential to the validity of the result. *C. Chevalley.*

Matsushima, Yozô. Sur les espaces homogènes kählériens d'un groupe de Lie réductif. Nagoya Math. J. **11** (1957), 53–60.

In this note the author proves the following theorem: Every compact homogeneous Kähler manifold is the cartesian product of a complex torus with Kähler metric and of a Kähler manifold which is homogeneous with respect to a compact, semi-simple Lie group, the Kähler metric of the cartesian product being the one naturally defined by those of the factors. This follows from another theorem of the same paper, namely: Let G/B be homogeneous Kählerian and let G be reductive and effective on G/B. Then B is compact, connected and is the centralizer in S of a torus of S, S being the semi-simple part of G. Moreover the center C of G and S/B both have invariant Kählerian structures, and G/B as a Kähler manifold is the cartesian product of these C and S/B.

These results extend parts of Proposition 1 and Theorem 1 of A. Borel, Proc. Nat. Acad. Sci. U.S.A. **40** (1954), 1147–1151 [MR **17**, 1108] and supply detailed proofs (the proofs being only sketched in the note cited).

W. M. Boothby (Evanston, Ill.).

Jongmans, F. Périodes des formes harmoniques attachées aux variétés kählériennes. Bull. Soc. Math. Belg. 1954, 24–34 (1955).

Jongmans, F. Etude des périodes des formes harmoniques. Bull. Soc. Math. France **83** (1955), 89–102.

A harmonic form of a Kähler variety is decomposable in a unique way into factors (primary forms) which are irreducible with respect to the unitary group, each factor being characterized by two integers, its class k and its type h [see, e.g., Guggenheimer, Comment. Math. Helv. **25** (1951), 257–297; MR **13**, 781].

The dimension of the space of harmonic forms of given class and type does not depend on the choice of Kählerian metric on the complex variety. The scalar product of harmonic forms of different class or type is equal to zero. Let π_u denote the periods of a primary harmonic form of order p with respect to a p-dimensional homology base. If $h \neq p/2$, then the following relations hold:

$$\sum A_{uv}\pi_u\pi_v = 0, \quad i^p \sum A_{uv}\pi_u\bar\pi_v = (-1)^{h+k}R_k,$$

where R_k is real >0 and A_{uv} are real (and, in the case of algebraic varieties, rational). From this it follows that the periods of a primary harmonic form with $h \neq p/2$ cannot be all real or all purely imaginary and also that

such a form is defined by giving the real parts of its periods. The generalized theorem of Poincaré is proved: if the matrix of periods of forms of given dimension has the structure $\begin{pmatrix} P_1 & O \\ P_3 & P_2 \end{pmatrix}$, where P_1 and P_2 are square matrices, then, by a change of basis of the forms and of the homology base, this matrix can be transformed into

$$\begin{pmatrix} P_1 & O \\ O & P_2 \end{pmatrix}.$$

A. M. Vasil'ev (RŽ Mat **1957**, no. 4349).

Couty, Raymond. Transformations définies par le groupe d'holonomie infinitésimale. C. R. Acad. Sci. Paris **244** (1957), 1871–1873.

The author continues here the study of the transformations $\mathcal{T}$ defined in C. R. Acad. Sci. Paris. **244** (1957), 553–555 [MR **18**, 931]. When the results are applied to pseudokählerian varieties, conditions are obtained for the Cartan tensor of such a variety to be null or symmetric. When the transformations $\mathcal{T}$ conserve the element of volume, equations are obtained analogous to the equations of Copson and Ruse for harmonic spaces. (Translation of the author's summary.) *E. T. Copson* (St. Andrews).

Tashiro, Yoshihiro. On a holomorphically projective correspondence in an almost complex space. Math. J. Okayama Univ. **6** (1957), 147–152.

Let X_{2n} have an almost complex structure $F_i{}^jF_j{}^h=-A_i{}^h$, and a symmetric affine connection $\Gamma_{ji}{}^h$ and let $F_i{}^j$ have covariant derivative 0. The solutions of

$$\frac{d^2x^h}{dt^2}+\Gamma_{ji}{}^h\frac{dx^j}{dt}\frac{dx^i}{dt}=\alpha(t)\frac{dx^h}{dt}+\beta(t)F_i{}^h\frac{dx^i}{dt}$$

are called holomorphically flat. The author introduces a "projective" curvature tensor $P_{kji}{}^h$ whose invariance is always necessary if these holomorphically flat curves are to correspond to one another. He then shows, as a sufficient condition, that for $P_{kji}{}^h=0$, the given space corresponds to a Euclidean one. If X_{2n} is a Kaehler space to start with, then the direct consequence from $P_{kji}{}^h=0$ is that the space has constant holomorphic curvature.

S. Bochner (Princeton, N.J.).

Tallini, Giuseppe. Sopra un teorema di A. Lichnerowicz sulla geometria Kähleriana. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **17** (1954), 204–209.

A new proof is given of a theorem of Lichnerowicz [Généralisations de la géométrie kählérienne globale. Colloque de Géométrie Différentielle, Louvain, 1951, pp. 99–122; MR **13**, 688]. If on a Riemannian manifold of even dimension there is given the field of a non-degenerate, skew-symmetric tensor F_{ik}, then there exists a positive definite metric g_{ik} satisfying the condition $F_{ij}F_{kp}g^{jp}=g_{ik}$, not necessarily coinciding with the originally given metric of the space.

A. M. Vasil'ev (RŽ Mat **1956**, no. 2447).

Dolbeault, Pierre. Formes différentielles méromorphes localement exactes. Trans. Amer. Math. Soc. **82** (1956), 494–518.

Let V be a compact complex-analytic manifold of complex dimension m, and let S be a non-singular complex-analytic subvariety of V of dimension $m-1$. Denote by $\mathcal{E}_S{}^{p+1,0}$ the complex vector space of classes of locally exact meromorphic $(p+1)$-forms on V having S as polar locus with multiplicity two at most, modulo the differentials of meromorphic p-forms on V having S as polar locus with multiplicity one at most. The problem investigated by the author is that of expressing $\dim \mathcal{E}_S{}^{p+1,0}$ as a function of the dimensions of certain spaces of cohomology of differential forms of class C^∞ on V and on S.

The paper is divided into two parts. The first part is concerned with the study of sheaves of germs of residues relative, on the one hand, to the sheaf $M_S{}^p$ (respectively $m_S{}^p$) of germs of meromorphic p-forms (respectively closed meromorphic p-forms) having S as polar locus with multiplicity one, and relative, on the other hand, to the sheaf $dM_S{}^p$. Finally, the dimensions of the cohomology spaces of V with values in these sheaves of germs of residues are evaluated (possibly under additional restrictions). More precisely, let Ω^p denote the sheaf of germs of holomorphic p-forms on V, $\Omega^p[S]$ the corresponding sheaf on the submanifold S. Denote by $\{S\}$ the complex line bundle over V corresponding to (the divisor) S, by $\{S\}_S$ the bundle over S induced from $\{S\}$ by the inclusion map $S\to V$, and let $\Omega^p(\{S\}_S)$ be the sheaf of germs of holomorphic p-forms on S with coefficients in $\{S\}_S$. Let E^p be the sheaf of germs of closed holomorphic p-forms on V, $E^p[S]$ the corresponding sheaf on S. If we let $A^{r,s}$ be the vector space of complex-valued differential forms on V of type (r, s) and if we write $B^p=\Sigma_{r,s\geq 0}A^{p+r,s}$ and denote by $K^{p,q}(V, \mathbf{C})$ the subspace of the $\bar\partial$-cohomology of B^p composed of elements of degree $p+q$, then we have the isomorphism $H^q(V, E^p)\cong K^{p,q}(V, \mathbf{C})$. Let $\mathcal{R}_S{}^p$, $r_S{}^p$ be the sheaves of residues of $M_S{}^p$, $m_S{}^p$ respectively; $r_S{}^p$ is canonically isomorphic to $E^{p-1}[S]$ and the following sequences are exact:

$$0\to\Omega^p\to M_S{}^p\xrightarrow{\bar\partial}\mathcal{R}_S{}^p\to 0,$$

$$0\to E^p\to m_S{}^p\xrightarrow{\bar\partial} r_S{}^p\to 0,$$

$$0\to\Omega^{p-1}[S]\to\mathcal{R}_S{}^p\to\Omega^p(\{S\}_S)\to 0.$$

In particular, $\dim H^q(V, r_S{}^p)=\dim K^{p-1,q}(S, \mathbf{C})$. Further, in the case where $\{S\}$ is sufficiently ample in the sense of Kodaira, $\dim H^q(V, \mathcal{R}_S{}^p)$ is expressed by a formula involving $\dim H^q(S, \Omega^p(\{S\}_S))$.

In the second part of the paper, the dimensions of the cohomlogy of V with values in $M_S{}^p$, $m_S{}^p$, $dM_S{}^p$ are calculated and used to prove the following theorem: Assume that V is a compact Kähler manifold and that $\{S\}$ is sufficiently ample (in the sense of Kodaira); then, if $p<m-3$, $\dim \mathcal{E}_S{}^{p+1,0}=\dim K^{p,1}(V, \mathbf{C})-\dim H^{p-1,0}(S, \mathbf{C})$ where $H^{p-1,0}(S, \mathbf{C})$ denotes the $\bar\partial$-cohomology of type $(p-1, 0)$ on the submanifold S. *D. C. Spencer.*

Blanchard, André. Sur les variétés analytiques complexes. Ann. Sci. Ecole Norm. Sup. (3) **73** (1956), 157–202.

The main problems studied in this paper are conditions under which a compact complex fiber space is a Kähler manifold or an algebraic variety. Let E be the fiber space, with base space B and fiber F. Let the fundamental group $\pi_1(B)$ operate trivially on the first real cohomology group $H^1(F)$. In order that E be Kählerian, the following conditions are necessary and sufficient: 1) There exists a Kähler form on F which represents a cohomology class invariant under $\pi_1(B)$; 2) B is Kählerian; 3) the transgression $H^1(F)\to H^2(B)$ in real cohomology is zero. It is this third condition, topological in nature, that is not fulfilled by known examples of non-Kählerian compact complex manifolds. As to the second question, suppose the fiber F connected, with a connected structural group. For E to be an algebraic variety the following conditions are

necessary and sufficient: 1) F and B are algebraic varieties; 2) $b^1(E)=b^1(F)+b^1(B)$; 3) the Albanese variety $A(E)$ of E is algebraic.

The paper is divided into two parts. The first treats of the automorphisms of a complex analytic manifold and the Albanese varieties. Let V be an algebraic variety, $A(V)$ its Albanese variety, and Φ a finite subgroup of the group of translations of $A(V)$. Then the main theorem of Part I states that if N_Φ is the inverse image of Φ in the connected group of automorphisms of V, then there exists an imbedding of V in a complex projective space such that N_Φ is induced by a group of collineations. Part II is mainly devoted to the proofs of the theorems stated in the last section. It also contains a proof of the theorem that if a compact Kähler manifold is blown up along a non-singular submanifold, then the new manifold is also Kählerian. *S. Chern* (Chicago, Ill.).

Holmann, Harald. Abbildungstheorie n-dimensionaler holomorph vollständiger Mannigfaltigkeiten. Schr. Math. Inst. Univ. Münster no. **10** (1956), i+59 pp.

Es sei X eine zusammenhängende n-dimensionale komplexe Mannigfaltigkeit, die eine kompakte reelle Liesche Gruppe G von Automorphismen zulässt, die alle einen festen Punkt P als Fixpunkt haben. G besitzt dann eine einparametrige Untergruppe G', deren Linearisierung bezüglich eines geigneten lokalen Koordinatensystems mit P als Nullpunkt von der Form $z_j \to z_j \cdot \exp(ip_j t)$ $(0 \leqq t \leqq 2\pi)$, ist; die p_j $(j=1, \cdots, n)$ sind dabei ganze Zahlen, deren grösster gemeinsamer Teiler gleich 1 ist. Die p_j sind nicht eindeutig bestimmt. Sie seien aber für das folgende festgewählt. Es werde vorausgesetzt, dass die p_j nicht negativ sind. Man numeriere sie so, dass $p_1, \cdots, p_r$ positiv sind und die weiteren p_j verschwinden. Ein Gebiet des C^r heisse $(p_1, \cdots, p_r)$-Gebiet, wenn es 0 enthält und durch die Gruppe $S=\{z_j \to z_j \cdot \exp(ip_j t)$ $(j=1, \cdots, r)$, $0 \leqq t \leqq 2\pi\}$ auf sich abgebildet wird. Es werde nun weiter vorausgesetzt, dass X eine Steinsche Mannigfaltigkeit ist. Dann gilt die Behauptung: Es existiert eine Steinsche Mannigfaltigkeit Y der Dimension $n-r$ und ein Vektorraumbündel W (Faser C^r) über Y, das den Zentralisator von S als Strukturgruppe zulässt (sodass also S in natürlicher Weise auf W und jeder Faser von W operiert), mit folgenden Eigenschaften: X kann eineindeutig und holomorph auf eine offene Teilmenge X' von W abgebildet werden und zwar so, dass der Durchschnitt von X' mit jeder Faser ein $(p_1, \cdots, p_r)$-Gebiet der Faser ist und dass jede Transformation aus G' in die entsprechende aus S übergeht.

Durch diesen Satz hat Verf. einen Abbildungssatz von H. Cartan [J. Math. Pures Appl. (9) **10** (1931), 1–114] von $n=2$ auf beliebiges n verallgemeinert. Für $n=2$ ist das Vektorraumbündel W immer trivial. Verf. führt eine Reihe allgemeiner Begriffsbildungen ein, in deren Rahmen er weitere Sätze beweist. Für die Beweise werden die Fundamentalsätze über kohärente analytische Garben [H. Cartan, Colloque sur les fonctions de plusieurs variables, Bruxelles, 1953, Thone, Liège, 1953, pp. 41–55; MR **16**, 235] und die Untersuchungen von H. Grauert [Math. Ann. **129** (1955), 233–259; C. R. Acad. Sci. Paris **242** (1956), 603–605; MR **17**, 80, 785] herangezogen. *F. Hirzebruch* (Bonn).

Grauert, Hans; und Remmert, Reinhold. Singularitäten komplexer Mannigfaltigkeiten und Riemannsche Gebiete. Math. Z. **67** (1957), 103–128.

Die Verff. stellen zunächst die wichtigsten Begriffe aus der Theorie der komplexen Räume zusammen. Als lokale Karten für komplexe Räume (im Sinne der Verff.) werden analytisch-verzweigte Überlagerungen eines Gebietes G des C^n genommen. (R, ϕ, G) heisst dabei analytisch-verzweigte Überlagerung, wenn R ein Hausdorffscher Raum ist, der durch ϕ stetig, eigentlich und nirgends-entartet in G abgebildet wird. „Nirgends-entartet" bedeutet, dass $\phi^{-1}(p)$ für jeden Punkt p von G diskret ist. Ferner wird verlangt, dass es eine analytische Menge A in G gibt, derart, dass $\phi^{-1}(A)$ den Raum R nirgends zerlegt und die Menge $R-\phi^{-1}(A)$ durch ϕ lokal-topologisch in G abgebildet wird. Verff. zeigen, dass für komplexe Räume gewisse Sätze aus der Theorie der komplexen Mannigfaltigkeiten nicht mehr gelten: I. Es gibt einen komplexen Raum Y_n, eine rein $(n-1)$-dimensionale analytische Menge E von Y_n und einen Punkt y_0 von E, derart, dass E in keiner Umgebung von y_0 als simultanes Nullstellengebilde von weniger als $(n-1)$ lokalen holomorphen Funktionen darstellbar ist $(n \geqq 3)$. II. Zwei rein $(n-1)$-dimensionale analytische Mengen in einem n-dimensionalen komplexen Raum können sich punkthaft schneiden. III. Es gibt komplexe Räume beliebiger Dimension mit meromorphen Funktionen, deren Unbestimmtheitsstellen isoliert liegen. Als Beispiel für I–III ziehen Verff. den Kegel Y_n über einer Segreschen Mannigfaltigkeit heran: Y_n liegt im C^{2n-2} (Koordinaten $z_1, w_1, z_2, w_2, \cdots, z_{n-1}, w_{n-1}$) und wird durch die Gleichungen $z_1/w_1=z_2/w_2=\cdots=z_{n-1}/w_{n-1}$ gegeben. Der Nullpunkt y_0 von C^{2n-2} ist der einzige nicht-uniformisierbare Punkt von Y_n $(n \geqq 3)$. Weiter zeigen die Verff., dass gewisse Sätze von Oka für komplexe Räume, die verzweigte Riemannsche Gebiete über dem C^n sind, für $n \geqq 3$ ihre Gültigkeit verlieren: IV. Es gibt endlich-blättrige, n-dimensionale Holomorphiegebiete, die aus lauter uniformisierbaren Punkten bestehen und mindestens einen nicht pseudo-konvexen Randpunkt besitzen $(n \geqq 3)$. V. Es gibt endlich-blättrige, n-dimensionale Holomorphiegebiete, die aus lauter uniformisierbaren Punkten bestehen und nicht holomorph-konvex sind $(n \geqq 3)$. *F. Hirzebruch* (Bonn).

★ **Hirzebruch, Friedrich. Der Satz von Riemann-Roch in Faisceau-theoretischer Formulierung: einige Anwendungen und offene Fragen.** Proceedings of the International Congress of Mathematicians, 1954, Amsterdam, vol. III, pp. 457–473. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam, 1956. $7.00.

This lecture describes some applications of the author's general form of the Riemann-Roch theorem. The topics treated are: (i) determination of the number $h^{p,q}$ of harmonic forms of type (p, q) on varieties which are complete intersections of hypersurfaces in projective space; (ii) properties of varieties whose Pontriagin classes of positive dimension are zero; (iii) remarks on the theory of the virtual arithmetic genus. A number of unsolved problems are mentioned. *J. A. Todd.*

Spanier, E. The homology of Kummer manifolds. Proc. Amer. Math. Soc. **7** (1956), 155–160.

L'Auteur appelle "Kummer manifolds" les variétés de dimension complexe n, obtenues à partir des variétés de Wirtinger [Monatsh. Math. Phys. **1** (1890), 113–128] en dé-singularisant les 2^{2n} points doubles coniques par des dilatations appropriées. Soit T le tore produit de $2n$ cercles. On entoure les images p_s des points singuliers de la variété W de Wirtinger par des boules B_s, et on considère la variété à bord $X=T-\bigcup B_s$. Soit G le groupe défini

par l'identité et l'opération $x \to x^{-1}$ (l'inverse étant défini par le groupe abélien sur T): la variété $X=\bar{X}/G$ qui a pour bord une somme d'espaces projectifs réels à $2n-1$ dimensions permet de construire la variété topologique de Kummer K. L'Auteur, après avoir démontré que K est simplement connexe, étudie la structure cohomologique, à coefficients dans les réels, les entiers, et les entiers modulo 2, successivement de $\bar{X}$, puis de X, et finalement de K. Le résultat est le suivant: La variété K est sans torsion; ce résultat avait déjà été pressenti par Andreotti, qui l'énonce maladroitement sur W [Univ. e Politec. Torino. Rend. Sem. Mat. **12** (1953), 239–264; MR **15**, 739]. Les nombres de Betti de K ont pour valeurs $r_i=0$ si i est impair, $=1$ si $i=0, 2n$, et $=\binom{2n}{i}+2^{2n}$ si i est pair $(0<i<2n)$.

L. Gauthier (Nancy).

See also: Functions of Complex Variables: Arens. **Partial Differential Equations:** Malgrange. **Differential Geometry:** Gugenheim and Spencer. **Manifolds, Connections:** Chern.

Algebraic Geometry

Godeaux, L. Sopra un fascio sovrabbondante di curve piane. Rend. Sem. Mat. Messina **1** (1955), 84–87.

Godeaux, Lucien. Remarques sur la formation des systèmes canonique et pluricanoniques de quelques surfaces algébriques. Acad. Roy. Belg. Bull. Cl. Sci. (5) **43** (1957), 226–234.

Godeaux, Lucien. Sur les points de diramation d'une surface multiple: points de seconde catégorie. Acad. Roy. Belg. Bull. Cl. Sci. (5) **43** (1957), 235–243.

Spampinato, Nicolò. Sulle curve osculatrici di un ramo lineare o superlineare di una curva algebrica. Rend. Accad. Sci Fis. Mat. Napoli (4) **23** (1956), 118–124 (1957).

Spampinato, Nicolò. Curve algebriche complete autoduali. Rend. Accad. Sci. Fis. Mat. Napoli (4) **23** (1956), 109–117 (1957).

Spampinato, Nicolò. Sui punti ipercomplessi di una ipersuperficie algebrica ordinaria dell' S_r complesso. Rend. Accad. Sci. Fis. Mat. Napoli (4) **23** (1956), 60–72 (1957).

Spampinato, Nicolò. Rappresentazione nell' S_{2r+1} complesso di una serie lineare, secata su una curva dell' S_r complesso da un fascio di ipersuperficie prolungata nel campo biduale. Rend. Accad. Sci. Fis. Mat. Napoli (4) **23** (1956), 17–32 (1957).

Spampinato, Nicolò. Carattere singolare e cuspidale di una curva storta completa. Rend. Accad. Sci. Fis. Mat. Napoli (4) **23** (1956), 10–16 (1957).

Galafassi, Vittorio Emanuele. Reti reali di curve piane algebriche a jacobiana priva di punti reali. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. **90** (1956), 378–382.

It is proved that the Jacobian curve of a real net of plane algebraic curves is free from real points if and only if: 1) each curve of the net consists, in the real field, of a single odd branch free from multiple points; 2) any two distinct branches of this type meet simply at only one point; 3) any two distinct real points of the plane lie on one and only one curve of the net.

As an easy consequence it is shown that a real plane algebraic curve, whose Hessian curve contains no real point, either is free from real points or, in the real field, consists of just one oval. This result was already obtained differently by H. Lewy [Amer. J. Math. **60** (1938), 555–560]. *B. Segre* (Rome).

Galafassi, Vittorio Emanuele. Forme reali armoniche. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. **90** (1956), 383–412.

The present paper deals with the harmonic forms, i.e., with the forms $f(x_0, x_1, \cdots, x_r)$ — of any degree n in a given number $r+1$ of variables — satisfying the Laplace equation $\Delta_2 f=0$, and with the harmonic hypersurfaces of order n of the projective space $S_r(x_0, x_1, \cdots, x_r)$, represented there by a corresponding equation $f=0$. The latter are the apolar forms of the tangential forms of class n containing the quadric $x_0^2+x_1^2+\cdots+x_r^2=0$ as a component, and have many interesting properties both in the real and in the complex domain. Thus, among other results, it is shown that every real harmonic hypersurface contains some real simple point and that, if $r\geqq 2$, any two real harmonic hypersurfaces of the same order of S_r meet at some real point.

Also, properties of linear systems of harmonic hypersurfaces are obtained, including a detailed study of the real pencils of harmonic plane curves, and the proof that, if $r\geqq 2$, the Jacobian of any ∞^r real linear system of harmonic hypersurfaces (of order $n\geqq 2$) in S_r contains some real point. This implies that, if $n\geqq 3$, $r\geqq 2$, the Hessian of any real harmonic hypersurface of order n of S_r contains some real point.

For $r=2$, the latter result had been firstly given quite differently by H. Lewy [Amer. J. Math. **60** (1938), 555–560]. The same result, and a number of the previous ones, had already been obtained for $r=1, 2$ with similar methods by B. Segre [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **15** (1953), 237–242, 339–344; MR **16**, 281]. *B. Segre* (Rome).

Galafassi, Vittorio Emanuele. Osservazioni sulla mia nota "Forme reali armoniche". Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. **90** (1956), 510–511.

It is pointed out that, if for $r>2$ there exist harmonic irreducible developable hypersurfaces of order $n>2$ (a question as yet undecided), some of the secondary results of the previous Note [see the paper reviewed above] have to be modified. No such hypersurface can be real, and so the modifications concern mainly the properties in the complex domain. *B. Segre* (Rome).

Stein, E. Product varieties of two Veroneseans; an example of generalised Segre varieties. Bull. Soc. Roy. Sci. Liège **26** (1957), 19–28.

Cette note généralise un travail antérieur [E. Stein, Nederl. Akad. Wetensch. Proc. Ser. A. **59** (1956), 104–107; MR **17**, 1131], consacré à la variété produit de deux courbes rationnelles normales. L'auteur considère ici le produit de deux variétés de Véronèse $V_m^{(h)}$ et $V_n^{(k)}$, dont les éléments générateurs ont respectivement les ordres h et k

dans les espaces linéaires $[m]$ et $[n]$, et qui appartiennent par suite à des espaces d'ordres $\left[\binom{m+h}{h}-1\right]$ et $\left[\binom{n+k}{n}-1\right]$. Il étudie la nature des correspondances ponctuelles entre les deux Véronésiennes ainsi que leurs paracollinéations, de même que la variété produit de $V_m^{(h)}$ et $V_n^{(k)}$ et sa représentation birationnelle. Il envisage ensuite les projections du produit de $V_m^{(h)}$ et de $V_n^{(k)}$, ce qui conduit à une construction de la variété de Segre de deux espaces linéairies $[m]$ et $[n]$ au moyen de couples $[V_m^{(h)}$ et $V_n^{(k)}]$ correspondant à des ordres h et k quelconques. L'exemple de la variété produit de deux surfaces de Véronèse illustre les résultats obtenus. *P. Vincensini* (Marseille).

Nollet, Louis. Les genres pseudocanoniques des surfaces algébriques régulières. Bull. Soc. Math. Belg. **8** (1956), 43–47.

Exposition rapide des résultats obtenus par l'A. sur les systèmes pseudocanoniques des surfaces algébriques régulières. Il s'agit d'une recherche sur la dimension des systèmes linéaires $|K+Z|$, où K est une courbe canonique de la surface considérée et Z est un diviseur algébrique de zéro. [Voir Nollet, Acad. Roy. Belg. Bull. Cl. Sci. (5) **42** (1956), 579–595; MR **18**, 234.] *E. Togliatti.*

Kirby, D. The structure of an isolated multiple point of a surface. I, II, III. Proc. London Math. Soc. (3) **6** (1956), 597–609; **7** (1957), 1–18, 19–28.

These three papers are an analysis of the structure of an algebraic surface V at a multiple (especially a double) point P, which is simple on a threefold U on which V lies. The technique used throughout is the study of the local equation of V in terms of a local coordinate system (x, y, z) on U, so that the results are essentially invariant under transformations of U regular at P.

In paper I, it is shown that if P is m-ple the equation of V can be written

$$\text{(i)} \qquad x^m+x^{m-2}g_2(y, z)+\cdots+g_m(y, z)=0,$$

where g_i is a formal power series of order $\geqq i$; and that for $m=2$, if

$$x^2+g_2(y, z)=0, \quad x'^2+g_2'(y', z')=0$$

are two possible local equations of V at P, then the series g_2, g_2' are equivalent under a regular transformation of (y, z) into (y', z') (equivalence including possible multiplication by a unit series). The branching sequence (i.e., sequence of multiple and proximate points) of a curve $x=g(y, z)=0$ (x, y, z being local coordinates and g a formal power series) is then shown to be essentially the same as in the classical case in which the coordinates are cartesian and g a polynomial. Paper I ends with the theorem that in the general case (i), P is a point of an m-ple curve of V if and only if there is a series $g(y, z)$ which is at least an i-ple factor of $g_i(y, z)$ for each i, $2\leqq i\leqq m$, $x=g(y, z)=0$ being then the equations of the m-ple curve.

Paper II is devoted to the case $m=2$, and shows that the branching sequence of the curve $x=g(y, z)=0$ completely determines the structure of V at P. In particular, if $g(y, z)$ is of order $s\leqq 3$, P gives rise only to isolated double points of the transformed surfaces after dilating P; but if $s>3$, there are $\frac{1}{2}(s-3)$ or $\frac{1}{2}(s-2)$ consecutive double lines, the last cuspidal or nodal according as s is odd or even, and in the last are points of the form $x^2=g^*(y, z)$, (or $x^2=zg^*(y, z)$ if s is odd), where $g^*(y, z)=0$ is the local equation of the plane curve, transform of $g(y, z)=0$, at one of the points arising from the dilation of the origin in the plane $x=0$. The local equations of V at a binode B_r, an ordinary unode U_{r+1}, and the exceptional unodes U_8^*, U_9^*, U_{10}^* are reducible to

$$x^2+y^2+z^r=0 \ (r\geqq 2), \quad x^2+y^2z+z^r=0 \ (r\geqq 3),$$

$$x^2+y^3+z^4=0, \quad x^2+y^3+yz^3=0, \quad x^2+y^3+z^5=0.$$

As the next simplest type of double point, hypernodes are then defined. In these, the curve $g(y, z)=0$ has a sequence of r triple points; V has the sequence: double point, double point, double line, repeated $\frac{1}{2}r$ or $\frac{1}{2}(r-1)$ times, — followed, if r is odd, by any kind of binode, or, if r is even, by either a conic node or a binode, or by nothing worse than pinch points. This paper concludes with a study of some situations analogous to these, in which the curve $g(y, z)=0$ has (i) m consecutive points of multiplicity $2n$, (ii) m consecutive points of multiplicity $2n$ with a simple branch through the first (with distinct tangent), and (iii) m consecutive points of multiplicity $2n+1$; and it is shown that for each of these there is a finite sequence of dilations, each of a point or a curve of the surface, which leaves at most a singularity for which the series $g(y, z)$ is of lower order than before.

Paper III returns to equation (i) and the case in which P is an m-ple point of V. It is shown by methods analogous to those of II, but more general and thus in a sense vaguer, that there is a finite sequence of dilations, determined by the branching sequences of the curves $g_i(y, z)=0$, each of which is either of an isolated m-ple point or of a non-singular m-ple curve of the surface, after which the transform of V has no point of multiplicity as high as m in the total image of all the neighbourhoods of P.

It follows from this that every algebraic surface lying on a non-singular threefold, and in particular every surface in S_1, can be entirely freed of singularities by a finite sequence of dilations of points and curves. It must be remembered, however, that there are many types of singular point on surfaces which are necessarily singular also for every threefold on which the surface lies — the general improper double point is an elementary example — and the present theory gives no information whatever about these.

{The reviewer wishes to take this occasion of protesting against the translation, latterly becoming current, of the Italian "dilatazione" by "dilatation". The plain English of this, in any dictionary, is "dilation", and there is no need for the barbarous neologism, which presumably would lead next to the verb "dilatate".} *P. Du Val.*

★ **Segre, Beniamino. Geometry upon an algebraic variety.** Proceedings of the International Congress of Mathematicians, 1954, Amsterdam, vol. III, pp. 497–513. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam, 1956. $7.00.

Verf. berichtet über einige neuere Entwicklungen in der Theorie der algebraischen Mannigfaltigkeiten (der Charakteristik 0). Es sei M eine singularitätenfreie algebraische Mannigfaltigkeit. Die Äquivalenzklassen algebraischer Zyklen bilden einen Ring. Verf. erwähnt die vier Äquivalenzbegriffe, die vor 20 Jahren von Severi eingeführt wurden (rational, algebraisch, topologisch, enumerativ); er verwendet die rationale Äquivalenz. Durch wiederholte Schnittbildung einer Hyperfläche A von M mit sich selbst erhält man die „kovariante Einbettungsfolge" von A in M. Wenn A eine Untermannigfaltigkeit der Codimension r ($r\geqq 2$) ist, dann kann man auf M die monoidale Transformation mit Basis A anwenden

und erhält die Mannigfaltigkeit M' und eine Hyperfläche A' von M'. Durch wiederholte Schnittbildung von A' mit sich selbst kann man eine kovariante Einbettungsfolge von A in M definieren. Ersetzt man M durch $M\times M$ und A durch die Diagonale von $M\times M$, dann führt dieser Prozess zu einer Definition der kanonischen Klassen von M [Segre, Ann. Mat. Pura Appl. (4) **37** (1954), 139–155; MR **16**, 511]. Ausgehend von den kanonischen Klassen bespricht Verf. den mit dem arithmetischen Geschlecht zusammenhängenden Fragenkomplex: Hilbert's charakteristische Funktion, Severi's Definition von p_a und P_a und seine Vermutung, dass $p_a=P_a=g_d-g_{d-1}+\cdots+(-1)^{d-1}g_1$, die Untersuchungen von Todd und schliesslich die neueren Resultate von Kodaira, Spencer und dem Ref., wodurch die Severische Vermutung und auch die Sätze von Todd, die noch von einer gewissen Hypothese abhingen, vollständig bewiesen wurden [für Literaturangaben siehe etwa Ref., Neue topologische Methoden in der algebraischen Geometrie, Springer, Berlin, 1956; MR **18**, 509]. Es gibt also 4 Definitionen für das arithmetische Geschlecht: p_a, P_a, alternierende Summe der g_i, Todd'sches Geschlecht (Linearkombination charakteristischer Zahlen). Verf. betont, dass die den charakteristischen Zahlen entsprechenden invarianten 0-dimensionalen algebraischen Zyklen (oder vielmehr ihre rationalen Äquivalenzklassen) mehr Beachtung finden sollten. Dann referiert Verf. noch kurz über die monoidalen Transformationen (und allgemeiner über die Modifikationen) komplexer Mannigfaltigkeiten, erwähnt die fast-komplexen Mannigfaltigkeiten und kommt schlieslich noch auf den fundamentalen Satz von Kodaira zu sprechen, dass jede Hodge Mannigfaltigkeit algebraisch ist. Der Bericht endet mit einigen Andeutungen über ungelöste Fragen. Interessant ist besonders Severi's Vermutung, dass jede algebraische Mannigfaltigkeit, für die alle kanonischen Klassen im Sinne rationaler Äquivalenz verschwinden, eine abelsche Mannigfaltigkeit ist. *F. Hirzebruch.*

Wallace, Andrew H. Algebraic approximation of manifolds. Proc. London Math. Soc. (3) **7** (1957), 196–210.

Let M be a compact differentiable manifold in euclidean n-space E. John Nash has shown [Ann. of Math. (2) **56** (1952), 405–421; MR **14**, 403] that M can be approximated by a sheet V of a real algebraic variety W in E. The present paper strengthens Nash's result by showing that V can be required to be isolated, in the sense that it has a neighborhood not intersecting any other sheet.

S. S. Cairns (Urbana, Ill.).

Lang, Serge. L-series of a covering. Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 422–424.

In this note the author makes use of the fact that the algebraic analogue of the theory of topological transformation groups is the theory of Galois coverings of algebraic varieties in order to draw some statements and conjectures about the behavior of the Artin L-series in such a Galois covering.

From every linear representation of a finite group an L-series is obtained. Let U, V be two algebraic varieties over a finite field with q elements and suppose that U is a Galois cover of V with finite group G. Let $H_1(U)$ denote the l-adic representation space of G arising from the Albanese variety $A(U)$ of U. Its dimension $B_1(U)$ equals $2g$, where g is the irregularity of U. We let F be the Frobenius transformation of U, and f the Frobenius transformation of V. The L-series obtained from this representation, it may be conjectured, should give exactly the contribution in the circle $|t|<q^{-(n-1)}$ to the L-series defined by the prime rational cycles of V. This extends the Weil conjectures on the zeta-function for varieties over a finite field [S. Lang and A. Weil, Amer. J. Math. **76** (1954), 819–827; MR **16**, 398]. *P. E. Conner.*

Lang, Serge; et Serre, Jean-Pierre. Sur les revêtements non ramifiés des variétés algébriques. Amer. J. Math. **79** (1957), 319–330.

U et V étant deux variétés algébriques définies sur un corps de caractéristique p quelconque, un revêtement de V est une application rationnelle partout définie $f: U\to V$ telle que: a) U et V soient normales (absolument irréductibles), de même dimension; b) pour tout $v\in V$, l'ensemble $f^{-1}(v)$ soit fini; c) la variété U soit complète au-dessus de tout point de V; d) si K et L sont les corps des fonction rationnelles sur V et sur U, l'extension L/K soit séparable.

Le degré n de l'extension L/K est appelé degré du revêtement f; on dit que U est non ramifié si, pour tout point $v\in V$, l'ensemble $f^{-1}(v)$ a exactement n éléments. Th. 1: Soit $f: U\to V\times W$ un revêtement non ramifié, V étant une variété complète; alors, il existe deux revêtements non ramifiés V' de V et W' de W tels que U soit un quotient du revêtement produit $V'\times W'$. Th. 2: Tout revêtement non ramifié d'une variété abélienne est une isogénie (separable) (la démonstration est calquée sur la démonstration classique prouvant que tout revêtement d'un group topologique est un groupe topologique). Th. 3: Soit V une variété normale, complète, définie sur un corps algébriquement clos k; alors, tout revêtement non ramifié de V est isomorphe à un revêtement défini sur k. Th. 4: Une variété normale et complète ne possède qu'un nombre fini de revêtements non ramifiés de degré donné; (on se ramène à démontrer le th. dans le cas où V est une courbe). Les ths. 1 et 4 généralisent des résultats classiques sur le corps des complexes; les ths. 2 à 4 sont démontrés à partir du th. 1 et la démonstration du th. 1 fait intervenir un lemme valable seulement pour les variétés complètes; les auteurs signalent que des contre-exemples (dus à la présence de „ramification supérieure") montrent que les conclusions des ths. 1 et 4 ne sont pas vraies sans restriction en caractéristique $p>0$; enfin, le th. 2 ne s'étend pas aux groupes algébriques non complets.

P. Dolbeault (Paris).

★ **Serre, Jean-Pierre. Cohomologie et géométrie algébrique.** Proceedings of the International Congress of Mathematicians, 1954, Amsterdam, vol. III, pp. 515–520. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam, 1956. $7.00.

In diesem Vortrag wurde gezeigt, wie die Theorie der Garben auch in der abstrakten algebraischen Geometrie (über einem Körper beliebiger Charakteristik) mit Erfolg angewendet werden kann. Inzwischen ist eine wichtige Arbeit des Verf. über diesen Gegenstand erschienen [Ann. of Math. (2) **61** (1955), 197–278; MR **16**, 953] deshalb kann Ref. sich hier kurz fassen.

Verf. skizziert u.a. einen Beweis für seinen Dualitätssatz im abstrakten Fall [für den klassischen Fall siehe Serre, Comment. Math. Helv. **29** (1955), 9–26; MR **16**, 736]. Weiter gibt Verf. eine Formel von Riemann-Roch für eine singularitätenfreie algebraische Mannigfaltigkeit X und einen Divisor D von X an, $\chi(X,\mathcal{L}(D))=\chi(X)-\chi x(-D)$; $\chi(X,\mathcal{L}(D))$ ist die Euler-Poincarésche Charakteristik von X bezüglich der Cohomologie (Zariski-Topologie) mit Koeffizienten in der Garbe $\mathcal{L}(D)$, deren Halme $\mathcal{L}(D)_x$ so definiert sind: Ein Element f von $\mathcal{L}(D)_x$ ist eine rationale Funktion f auf X, deren Divisor (f) die Un-

gleichung $(f)+D\geqq 0$ erfüllt. $\chi(X)=\chi(X, \mathscr{L}(0))$ ist das arithmetische Geschlecht von X und $\chi_X(-D)$ das virtuelle arithmetische Geschlecht des Divisors $-D$. Unter Verwendung des Dualitätssatzes kann Verf. die Vermutung von Severi beweisen, dass $p_a(X)=P_a(X)=(-1)^m(\chi(X)-1)$, wo m die Dimension von X ist [für den klassischen Fall siehe Kodaira und Spencer, Proc. Nat. Acad. Sci. U.S.A. **39** (1953), 641–649; MR **16**, 617]. Ungelöst bleibt, ob $\chi(X, \mathscr{L}(D))$ durch ein "Polynom" in D und den kanonischen Klassen von X dargestellt werden kann; das wäre der Satz von Riemann-Roch, wie er für den klassischen Fall vom Ref. bewiesen wurde [Neue topologische Methoden in der algebraischen Geometrie, Springer, Berlin, 1956; MR **18**, 509]. Verf. stellt dann noch Fragen bezüglich der von ihm definierten Zahlen $h^{p,q}(X)$. Untersuchungen von J. Igusa [Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 964–967; MR **17**, 534] haben gezeigt, dass sich diese Zahlen anders verhalten als im klassichen Fall, nämlich $h^{1,0}(X)$ braucht nicht gleich der Dimension der Picardschen Mannigfaltigkeit von X zu sein.

F. Hirzebruch (Bonn).

Gomboli, Giuliana. Estensione di alcuni classici teoremi di geometria sopra la curva al caso di un campo base arbitrario. Rend. Mat. e Appl. (5) **15** (1956), 315–328 (1957).

L'A. studia alcune proprietà di geometria sopra una curva usando i metodi dell'algebra astratta [cf. C. Chevalley, Introduction to the theory of algebraic functions of one variable, Math. Surveys, no. 7, Amer. Math. Soc., New York, 1951; MR **13**, 64]. Con riferimento ad un campo di funzioni algebriche di una variabile, su un campo base arbitrario, l'A. dimostra anzitutto l'equivalenza tra il genere g definito da Chevalley e il genere p definito nel modo classico da Weierstrass (ciò allo scopo di chiarire l'equivalenza tra il teorema di Riemann-Roch classico e quello in forma moderna, utilizzato nelle successive dimostrazioni). Segue una dimostrazione del teorema di riduzione diversa da quella già data da Chevalley. La parte principale del lavoro contiene l'estensione dal caso classico del teorema delle lacune di Weierstrass e del teorema di Clifford. Il teorema delle lacune viene poi ulteriormente approfondito nel caso di un campo base algebricamente chiuso. *M. Rosati* (Roma).

★**Néron, A. Propriétés arithmétiques de certaines familles de courbes algébriques.** Proceedings of the International Congress of Mathematicians, 1954, Amsterdam, vol. III, pp. 481–488. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam, 1956. $7.00.

The following theorem is proved. Let A, B be two abelian varieties, u a homomorphism of A on B, W a non-abelian subvariety of A such that W and B have the same dimension and B is the support of $u(W)$. Moreover, suppose that A, B, W, u are defined upon the same finite algebraic field k, and that B contains an infinity of rational points on k not lying on a proper abelian subvariety of B. Now, if h is any real number, denote by $\varphi(h)$ the number of points M of B such that the intersection $u^{-1}(M)\cdot W$ is defined and has a "highness" [according to A. Weil, Ann. of Math. (2) **53** (1951), 412–444; MR **13**, 66] or a "complexity" [according to Northcott, Proc. Cambridge Philos. Soc. **45** (1949), 502–509; MR **11**, 390] lower than h; next, denote by $\psi(h)$ the number of the above points M such that one of the components of $u^{-1}(M)\cdot W$ is rational on k. Then, for any real number $\varepsilon>0$, the inequality $\psi(h)<e\varphi(h)$ holds for all sufficiently large values of h.

This result is similar to the famous irreducibility theorem [D. Hilbert, J. Reine Angew. Math. **110** (1892), 104–129] concerning the case when A is any algebraic variety and B is a linear space. By using it suitably in the case when B is the representative variety of a system of algebraic curves, the author establishes the existence of curves of a given genus g and of high rank on the rational field, and precisely of rank 11 for $g=1$ and of rank $3g+7$ for $g\geqq 2$. These results on the rank improve by unity the results the author himself had previously deduced [Bull. Soc. Math. France **80** (1952), 101–166; MR **15**, 151], in a similar manner, from Hilbert's theorem. *B. Segre.*

★**Rosenlicht, Maxwell. Group varieties and differential forms.** Proceedings of the International Congress of Mathematicians, 1954, Amsterdam, vol. III, pp. 493–496. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam, 1956. $7.00.

The author shows how abstract analogues of certain results concerning differential forms with singularities, in particular simple differentials of the second kind, can be given in terms of noncomplete commutative algebraic group varieties. To facilitate this, he defines a differential to be of the second kind if at each place it can be expressed as the sum of an exact differential and a differential that has no pole at the place. He also develops a related geometric theory in terms of fiber bundles for differentials of the second kind valid for any characteristic. Finally, additional problems of a similar nature are suggested.

T. R. Hollcroft (Aurora, N.Y.).

★**Rosenlicht, Maxwell. Commutative algebraic group varieties. Algebraic geometry and topology.** A symposium in honor of S. Lefschetz, pp. 151–156. Princeton University Press, Princeton, N. J., 1957. $7.50.

This paper contains statements (mostly without proofs) of various theorems on algebraic groups. Among these, the most important one seems to be to the effect that, if an algebraic group G operates regularly on a variety V, then there exists a local ring in the field F of functions on V whose field of quotients is F and which is stable under the operations of G. The author shows how it is possible to deduce from this fact an elementary proof of the theorem which says that a commutative algebraic group contains a normal linear subgroup whose factor group is an Abelian variety. *C. Chevalley* (Paris).

Germer, Henning. Einige kubische und quadratische Transformationen im projektiven R_n. Math. Nachr. **15** (1956), 299–338.

Let $x^1=xA$ and $xKy'=f(x, y)=0$ be a collineation and a correlation, respectively, which are defined in the same n-dimensional projective space; A and K are matrices over the complex field. The author considers the Cremona correspondence (*) $y=x^1f(x, x)-xf(x, x^1)$, where y is the point of intersection of the straight line xx^1 with the hyperplane $f(x, y)=0$ that corresponds to x by the correlation. Several types of (*) and of its inverse correspondence are discussed, taking into account the distinct projective types of the collineation and its relation with the correlation. The author obtains the fundamental subvariety of the correspondence (*) and proves the existence of the distinct types of (*) that he has considered. *P. Abellanas* (Madrid).

See also: Geometries, Euclidean and other: Bereis and Brauner. **Complex Manifolds:** Blanchard; Hirzebruch.

NUMERICAL ANALYSIS

See: **Foundations, Theory of Sets, Logic:** Markov.

Numerical Methods

Samokiš, B. A. Investigation of the rapidity of convergence of the method of quickest descent. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 1(73), 238–240. (Russian)

Nonweiler, T. A method for the numerical evaluation of an integral (occurring in the expression for the wave drag of slender bodies). Coll. Aero. Cranfield. Rep. no. **100** (1956), 24 pp.

Il s'agit de l'intégration numérique de l'intégrale

$$\int_0^1\int_0^1 F(x,y)\ln\left|\frac{1}{x-y}\right|dxdy.$$

L'auteur utilise le développement en séries double de Fourier avec $\theta=\cos^{-1}(1-2x)$ et $\varphi=\cos^{-1}(1-2y)$. On rencontre ce type d'intégrale dans l'aérodynamique des vitesses supersoniques. *M. Kiveliovitch* (Paris).

★ **Bauer, F. L. Beiträge zum Danilewski-Verfahren.** Aktuelle Probleme der Rechentechnik. Bericht über das Internationale Mathematiker-Kolloquium, Dresden, 22. bis 27. November 1955, pp. 133–139. VEB Deutscher Verlag der Wissenschaften, Berlin, 1957.

Danilewski's method to determine the characteristic polynomial $x^n+\sum_{r=1}^n b_r x^{n-r}$ of a given $n\times n$-matrix A consists in finding a matrix T such that TAT^{-1} is the rational normal form

$$\begin{pmatrix} 0 & & 0 & -b_n \\ 1 & \ddots & \vdots & \vdots \\ & \ddots & 0 & -b_{n-1} \\ 0 & & 1 & -b_1 \end{pmatrix}=A^{(n)}.$$

The similarity represented by T is obtained by a succession of similarities $A^{(i+1)}=T_iA^{(i)}T_i^{-1}$ $(i=1,\cdots,n-1, A^{(1)}=A)$. For the matrices T satisfying this condition a slightly modified form, namely

$$T_i^{-1}=\begin{pmatrix} 0' & t_i \\ I & t_i' \end{pmatrix}$$

is proposed here by which the method should be more convenient for application in automatic computation. In a slightly different appearence the method is stated to coincide with a process described by Krylov, Lusin and Frazer-Duncan-Collar, where

$$T^{-1}=(t\ At\ A^2t\cdots A^{n-1}t)$$

The influence of the choice of the initial vector t on the numerical quality of the result is discussed and some extensions and applications are described in general terms.
H. Schwerdtfeger (Montreal, P.Q.).

Brooker, R. A.; and Sumner, F. H. The method of Lanczos for calculating the characteristic roots and vectors of a real symmetric matrix. Proc. Inst. Elec. Engrs. B. **103** (1956), supplement no. 1, 114–119.

The purpose of this present paper is to call attention to the method for finding eigenvalues and eigenvectors of a real symmetric matrix, that was introduced by C. Lanczos [J. Res. Nat. Bur. Standards **45** (1950), 255–282; MR **13**, 163]. A detailed description of the method is given together with a description of the devices introduced by Lanczos in order to separate nearly equal eigenvalues. The experimental results verify the effectiveness of the method. *M. R. Hestenes* (Los Angeles, Calif.).

Barrett, W. On approximate factors of polynomials. Quart. J. Math., Oxford Ser. (2) **6** (1955), 293–300.

Die Arbeit knüpft an eine Abhandlung des Referenten an [Acta Math. **72** (1940), 99–155, 157–257 (im folgenden zitiert mit M. G.); MR **1**, 323; **2**, 342]. Ist $A(x)=\sum_{\nu=0}^n a_\nu x^\nu$, $a_0a_n\neq0$, ein Polynom n-ten Grades und betrachtet man das Polynom $A_0(x)=a_m+a_{m+1}x+\cdots+a_{m+k}x^k$, das aus einer Koeffizientensequenz von $A(x)$ gebildet wird, so wurde in M. G. nachgewiesen, daß unter gewissen Bedingungen $A(x)$ einen Teiler k-ten Grades $C(x)=\sum_{\nu=0}^k c_\nu x^\nu$ besitzt, dessen Koeffizienten sich „wenig" von denen von $A_0(x)$ unterscheiden. Die Formulierung dieser Bedingungen benutzt wesentlich den Begriff der Deviationen D_k, die in M. G. mit Hilfe des Newtondiagramms eingeführt wurden, sich aber auch direkt durch die Formel

$$D_k=\mathrm{Max}\left[1,\ \min_{\mu,\nu>0}\left(\left|\frac{a_k}{a_{k-\nu}}\right|^{1/\nu}\cdot\left|\frac{a_k}{a_{k+\mu}}\right|^{1/\mu}\right)\right]$$

darstellen lassen (dieser Ausdruck für D_k ist in der Arbeit des Verf. durch Druckfehler entstellt). In M. G. wurde nun gezeigt, daß, wenn $M=\mathrm{Min}\,(D_m, D_{m+k})\geqq18{,}7$, bzw. im Falle $m+k=n$, $M=D_m\geqq13{,}5$ ist, obige Zerlegungssatz gilt und man zugleich hat: $C(x)-A_0(x)\ll\delta\mathfrak{R}_{A_0}(x)$, wo $\mathfrak{R}_{A_0}(x)$ die sog. Newtonsche Majorante von $A_0(x)$ ist, während δ in beiden Fällen von M abhängt und für $M\to\infty$ im ersten Fall $=2/M+O(1/M^2)$, im zweiten $=1/M+O(1/M^2)$ ist. Für diesen Satz skizziert Verf. eine neue Beweismethode, die auf dem Brouwerschen Fixpunktsatz beruht. Diese Methode liefert zugleich eine gewisse Verbesserung der Konstanten, indem 18,7 durch 10,8 und 13,5 durch 9 ersetzt werden kann. Zugleich deutet der Verf. an, daß 9 für den Fall $m+k=n$ die „beste" Konstante ist. Er scheint auch anzugeben, daß die Werte von δ sich durch seine Methode als kleiner ergeben, doch überblickt man leicht, daß die von ihm angegebenen Ausdrücke in den beiden Fällen gleichfalls die oben angegebenen asymptotischen Darstellungen besitzen. Obgleich der obige Zerlegungssatz im Rahmen der Graeffeschen Methode von vornherein für relativ sehr große Werte von M angewandt wird, so wäre dennoch die Verbesserung der beiden angegebenen Konstanten nicht nur (im Falle von 9) von prinzipieller, sondern auch von praktischer Bedeutung, da ihre Verwendung gelegentlich ein paar Rechengänge ersparen kann und manchmal etwas früher das „Einsetzen der Konvergenz" zu konstatieren gestattet. Die Andeutungen über dei Beweise sind allerdings sehr lückenhaft, und es wäre zu hoffen, daß Verf. an Hand dieser Andeutungen seinen Beweis vollständig rekonstruiert und ihn in einer Weise darstellt, die dem Leser nur normalen Aufwand an Zeit und Arbeit zumutet.
A. Ostrowski (Zbl. **66** (1957), 269).

★ **Varoli, Giuseppe. Sulla determinazione, col metodo di iterazione, del tasso di una rendita periodica a termini variabili in progressione geometrica.** Scritti matematici in onore di Filippo Sibirani, pp. 289–325. Cesare Zuffi, Bologna, 1957.

Each of the two equations $v(1-q^nv^n)=A(1-qv)$ and

$r^n-q^n=S(r-q)$ with unknown $r=1+i=v^{-1}$ is written in a variety of forms for iterative solution, with detailed discussion of conditions for convergence.

A. S. Householder (Oak Ridge, Tenn.).

Hammer, Preston C.; and Wymore, A. Wayne. Numerical evaluation of multiple integrals. I. Math. Tables Aids Comput. **11** (1957), 59–67.

This paper deals with various aspects of the complicated and important problem of multiple integration. First several theorems are given about integrals over regions which are transforms of simpler regions and over Cartesian product regions. Then the authors specialize their discussion to symmetrical regions and illustrate their approach with an example which shows that, even over these regions, it is very difficult to derive high-order formulae. Finally, one approach to error analysis is briefly touched on. The authors promise a sequel in which they will present particular formulas for various symmetrical regions.

P. Rabinowitz (Rehovoth).

Zuber, R. Graphical solution of ordinary differential equations of the first order. Zastos. Mat. **3** (1956), 82–89. (Polish. Russian and English summaries)

The author gives two graphical methods analogous to the isocline method. Suppose $F(x, y, y')=0$ is given. The methods consist of taking a curve (1) $x=a(t)$, $y=b(t)$, and for some set of values of t drawing either curve (2) $F(x, y, (y-b(t))/(x-a(t)))=0$ or (3) $F(x, y, -(x-a(t))/(y-b(t)))=0$. Elements of an integral curve are constructed by drawing line segments through points of (2) and the corresponding points $P_0(a(t), b(t))$ of (1), or drawing arcs of circles through points of (3) with their centers at P_0. The curve (2) is the locus of points whose line elements pass through P_0, and the curve (3) is the locus of points whose line elements are normal to the lines passing through P_0. The author claims that in general these methods are more convenient than the isocline method. In the opinion of the reviewer even the examples given by the author (e.g. Fig. 1. and Fig. 2) show that the methods proposed are more difficult to carry out than the isocline method. The only advantage of the second method is the possibility of approximating an integral curve graphically by the arcs of osculating circles instead of tangent lines segments.

C. Masaitis.

★ **Unger, H. Matrizenverfahren bei linearen Differentialgleichungsproblemen.** Aktuelle Probleme der Rechentechnik. Bericht über das Internationale Mathematiker-Kolloquium, Dresden, 22. bis 27. November 1955, pp. 141–149. VEB Deutscher Verlag der Wissenschaften, Berlin, 1957.

Given is the differential equation (1) $L(y)=r(x)$ where $L(y)\equiv y^{(n)}-\sum_{\nu=0}^{n-1} f_\nu(x)y^{(\nu)}$ with the column of initial values

$$\mathfrak{y}(a)=\mathfrak{y}_a=\begin{pmatrix} y(a) \\ y'(a) \\ \vdots \\ y^{(n)}(a) \end{pmatrix}$$

and the boundary conditions $\mathfrak{R}_a\mathfrak{y}_a+\mathfrak{R}_b\mathfrak{y}_b=\mathfrak{k}$. By repeated differentiation and by using

$$(1)\qquad y^{(n+p)}=\sum_{\nu=0}^{n-1}\alpha_{n+p,\nu}y^{(\nu)}+r_{n+p}$$

the Taylor series $y(x)=y(\xi)+hy'(\xi)+\cdots$ $(h=x-\xi)$ is turned into $y(x)=\sum_{\nu=0}^{n-1} a_{1,\nu}(\xi, h)y^{(\nu)}(\xi)+g_1(\xi, h)$; thus, by $n-1$ differentiations,

$$(2)\quad \mathfrak{y}(x)=\mathfrak{A}(\xi, h)\mathfrak{y}(\xi)+\mathfrak{y}(\xi, h),\ \mathfrak{A}=(a_{i\nu})=\frac{\partial^{i-1}a_{1\nu}}{\partial h^{i-1}}$$

$(i=1, 2, \cdots, n)$. This matrix may be found approximately by subdividing the basic interval $\langle a, b\rangle$ by the points $x_1, \cdots, x_n$ for which numerical values of y are to be computed. In a similar manner the eigenvalue problem $L(y)=\lambda M(y)$ can be reduced into a linear system with parameter λ: $\mathfrak{y}(x)=\mathfrak{A}(\xi, h, \lambda)\mathfrak{y}(\xi)$. A reduction of this kind is also mentioned for a system of linear differential equations. References to the literature are given. All these methods are valid only within an interval of unrestricted regularity of all the functions concerned. Finally there is a discussion of applications to engineering problems.

H. Schwerdtfeger (Montreal, P.Q.).

Adachi, Ryuzo. A method on the numerical solutions of some differential equations. Kumamoto J. Sci. Ser. A. **2** (1955), 244–252.

The author considers ordinary differential equations like $y''=f(x, y, y')$ under the general conditions $g_j(y_0, \cdots y_m; y_0, \cdots y_m')=0$ $(j=1, 2)$ where $y_i=y(x_i)$ and $x_i=a+ih$ $(i=0, 1, \cdots, m)$. Numerical solutions are obtained by taking differences for the derivatives. The differences refer to the values y_i; they represent higher order approximations. The difference equations are solved by iteration. Some numerical examples of nonlinear differential equations are presented in which the functions g_j are linear.

H. Bückner (Schenectady, N.Y.).

John, Fritz. Numerical solution of the equation of heat conduction for preceding times. Ann. Mat. Pura Appl. (4) **40** (1955), 129–142.

The author discusses solution of

$$(*)\qquad \frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}\ (-\infty<x<+\infty,\ -T\leqq t\leqq 0)$$

$$u(x, 0)=f(x)$$

in cases where the solution exists, by the expression

$$(**)\qquad \bar{v}(x, -t)=\sum_{j=-m}^{+m} c_j^m(t)\bar{f}(x+jh)$$

$\{\bar{f}(x+jh)\}$ is an approximation to $\{f(x+jh)\}$; $\{c_j^m(t)\}$ is chosen so that $v(x, -t)$, obtained from (**) by replacing $\bar{f}$ by f, is an exact solution of (*) whenever $f(x)$ is a polynomial of degree $\leqq 2m$ (i.e., the truncation error is zero). By very careful analysis, the "truncation error" $(u-v)$ and the "data error" $(v-\bar{v})$ are estimated. The paper concludes with certain tables of $\{c_j^m\}$ and related coefficients.

M. A. Hyman (Philadelphia, Pa.).

Batschelet, Eduard; und Grün, Franz. Numerische Behandlung der Diffusionsgleichung mit Konvektionsterm. Z. Angew. Math. Phys. **7** (1956), 113–120.

In this paper a diffusion problem is considered originating from a study on gas exchange in a lung capillary. The differential equation is $D\Delta(r, x)u-v(r)u_x=0$, $r<R$, $x>0$ with the boundary conditions: $x=0$, $u=u^*$; $r=R$, $u_r=-ku$. The numerical treatment consists in replacing the differential equation by a set of difference equations which are solved exactly, thus avoiding difficulties inherent to the relaxation method. A numerical example for which $v=v_0(1-r^2/R^2)$ is treated in detail. This case, however, could have been solved analytically by introduction of Laguerre functions. The truncation error is discussed only summarily.

H. A. Lauwerier.

Stiefel, E. On solving Fredholm integral equations. Applications to conformal mapping and variational problems of potential theory. J. Soc. Indust. Appl. Math. **4** (1956), 63–85.

The present paper is concerned with the solution of the linear equation $Ax=h$, where A is a linear operator in a space R, by means of iterations of the type

$$x_n=\alpha_1 k+\alpha_2 Ak+\cdots+\alpha_n A^{n-1}k.$$

The residual is

$$r_n=k-Ax_n=R_n(A)k$$

where $R_n(\lambda)$ is the polynomial

$$R_n(\lambda)=1-\alpha_1\lambda-\alpha_2\lambda^2-\cdots-\alpha_n\lambda^n.$$

The type of iteration depends upon the choice of $R_n(\lambda)$. The important case in which the polynomials are orthogonal in the interval $a\leqq\lambda\leqq b$ is discussed and in particular the case of Tchebycheff polynomials is discussed. The results are applied to the problem of finding the Green's function for the two dimensional case. The method is applicable also in the case of higher dimensions. The special cases of nearly circular regions and elongated regions are considered in particular. It is shown that these iterations can be derived by a variational approach. The results obtained are applied to the problem of Neumann. The methods here described are applicable to nearly singular problems. *M. R. Hestenes.*

Seebeck, C. L., Jr.; and Hoelzer, H. Best fitting integral curves of linear differential equations. Amer. Math. Monthly **64** (1957), 348–351.

Given a differential equation

(*) $$b_0y+b_1y'+\cdots+b_my^{(m)}=D(x),$$

where $D(x)$ is a function such that $\overline{M}_k=\int_0^\infty x^kD(x)dx$ are finite constants not all zero for all $k=0, \cdots, 2m$, and the b_k are disposable parameters. Let $y=F(x)$ be a function with this same property that $M_k=\int_0^\infty x^kF(x)x$ dexist for $k=0, \cdots, 2m$. Theorem: If there exists a set of numbers b_k such that $y=F(x)$ is a solution of (*), then this set is uniquely determined by M_k and $\overline{M}_k$, provided the determinant $|C_{i,j}|\neq 0$, where $i=1, \cdots, m+1$, $j=1, \cdots, m+1$ and $C_{ij}=C_k=(-1)^kM_k/k!$ $(k=i+j-2)$. This result enables us to determine such coefficients and initial conditions for the given equation (*) that its integral curve fits a given set of m tabular data in a manner very suitable for use in an analogue computer.

K. Chandrasekharan (Bombay).

Bonfiglioli, L. A nomographic method of graphic integration. Bull. Res. Council Israel. Sect. A. **6** (1956), 1–10.

The proposed nomographic method of graphic integration is based on the representation of each ordinate of a given curve $y=f(x)$ as the tangent of the angle between its respective ray and the x-axis. The tangent values are marked on a nomographic scale of the function $qy\sqrt{2/(y+1)}$, and the points thus obtained connected by rays to the origin of the nomogram. Integration is carried out by means of a strip of tracing paper with three parallel straight lines drawn on it. The strip is moved over the nomogram and the segments cut off by the rays are added up graphically on the third line. *Author's summary.*

Blaschke, Wilhelm. Sui problemi fondamentali della Nomografia. Rend. Mat. e Appl. (5) **15** (1956), 46–52.

Employing the geometric methods of Frobenius and Cartan, conditions for the representations of functions by nomograms of genus zero are given. *R. T. Herbst.*

Ostrovskiĭ, G. M. Graphical integration of certain nonlinear equations of the theory of oscillation. Vestnik Moskov. Univ. **11** (1956), no. 5, 25–30. (Russian)

Simple geometric procedures are described for constructing the direction field for differential equations of the form $vdv+[f(x)v+g(x)]dx=0$, which arise naturally in vibration problems (with $v=dx/dt$). Extensions are indicated to equations arising from forced vibrations.

W. Kaplan (Ann Arbor, Mich.).

★ **Kotani, Masao; Amemiya, Ayao; Ishiguro, Eiichi; and Kimura, Tôsaku. Table of molecular integrals.** Maruzen Co., Ltd., Tokyo, 1955. vi+230 pp. $7.50.

The volume collects together corrected versions of previously published tables of integrals occurring in the theory of molecular structure and the associated auxiliary functions [MR **2**, 239; **15**, 352; **16**, 287]. It includes also some values which have not appeared previously .The molecular integrals are of the one and two-centre types involving $1s$, $2s$, $2p\sigma$ and $2p\pi$ Slater orbitals and are particularly useful for the consideration of homonuclear and diatomic hydride molecules. There is a useful summary of the quantum mechanics of many electron systems which includes a table of matrices of irreducible representations of the symmetric permutation group, and there is a detailed description of the analytical and numerical methods by which the integrals were obtained, a few formulae for integrals involving $3s$, $3p$ and $3d$ orbitals being included. These tables together with the recent compilation by H. Preuss [Integraltafeln zur Quantenchemie, Bd. 1, Springer, Berlin, 1956] provide substantially all the available values of molecular integrals. *A. Dalgarno* (Belfast).

Ishiguro, Eiichi; Arai, Tadashi; and Sakamoto, Michiko. Tables useful for the calculation of the molecular integrals. IX. Nat. Sci. Rep. Ochanomizu Univ. **6** (1956), 204–214.

[For parts I–VIII see same Rep. **1** (1951), 22–28; **2** (1951), 34–42; **3** (1952), 53–62; **4**, 64–76 (1953), 176–191 (1954); **5**, 33–58 (1954), 197–212 (1955); **6** (1955), 157–181; MR **14**, 799; **15**, 255; **16**, 175, 750; **17**, 542, 1167.] Various one-centre and two-centre molecular integrals involving combinations of $1s$, $2s$, $2p\sigma$ and $2p\pi$ Slater orbitals with nuclear screening parameters δ_1 and δ_2 are tabulated for the pairs of values $(\delta_1R, \delta_2R)=(14.00, 3.00)$, (13.25, 3.25), (14.00, 3.50) and (17.75, 5.25), R being the nuclear separation. The tables differ from those of Kotani et al. [see the paper reviewed above] in that orthogonalized $2s$ orbitals are used. *A. Dalgarno* (Belfast).

Alder, Kurt; and Winther, Aage. Tables of the classical orbital integrals in Coulomb excitation. Mat.-Fys. Medd. Danske Vid. Selsk. **31** (1956), no. 1, 74 pp.

In the classical description of the Coulomb excitation process one is led to some non-elementary integrals of the form

$$\int_{-\infty}^{\infty} e^{i\xi(e\sinh w+w)}\left\{\frac{[\cosh w+e+i(e^2-1)^{\frac{1}{2}}\sinh w]^\mu}{(e\cosh w+1)^{\lambda+\mu}}\right\}dw,$$

where $(1/e)=\sin(\theta/2)$. These integrals can be used also in a more refined treatment, related to the WKB method, for the calculation of the quantum mechanical corrections. This paper contains a collection of formulas, giving mathematical properties and asymptotic expressions for the integrals. An explicit tabulation of the integrals for a large range of the parameters is given. *P. M. Morse.*

See also: Linear Algebra: Greville. **Geometry of Numbers:** Frank. **Sequences, Series, Summability:** Wynn; Obreschkoff. **Approximations, Orthogonal Functions:** Vertgeim. **Partial Differential Equations:** Gould. **Difference Equations, Functional Equations:** Evangelisti. **Topological Vector Spaces:** Gel'fand. **Banach Spaces, Banach Algebras, Hilbert Spaces:** Altman. **Computing Machines:** Ting Kuan. **Probability** Nicholson. **Mechanics of Particles and Systems:** Kosticyn. **Quantum Mechanics:** Kaempffer and Kennedy. **Geophysics:** Levallois.

Computing Machines

★**Kämmerer, W. Die programmgesteuerte Relais-Zwillings-Rechenanlage "Oprema" des VEB Carl Zeiss, Jena.** Aktuelle Probleme der Rechentechnik. Bericht über das Internationale Mathematiker-Kolloquium, Dresden, 22. bis 27. November 1955, pp. 15–16. VEB Deutscher Verlag der Wissenschaften, Berlin, 1957.

Halstead, M. H.; Richman, Robert L.; Covey, Winton; and Merryman, Jerry D. A preliminary report on the design of a computer for micrometeorology. J. Meteorol. **14** (1957), 308–325.

Gaudfernau, Liliane. Méthode de résolution autocommandée sur calculateur électronique arithmétique. Rech. Aéro. no. **54** (1956), 31–38.

The author first outlines the principal features of an automatic digital computer and then describes iterative and other processes to determine an unknown. Preferring the simpler, digit by digit methods, he considers the solution of $f(z)=0$ where the roots may be either real or complex. Finally, a number of related problems are mentioned, including a method for the solution of a differential equation, and concluding with some general remarks. *D. C. Gilles* (Manchester).

Ting Kuan, Shu-chuang. A simple device for approximating solution of dynamic systems occurring in economics. Metroecon. **8** (1956), 186–198.

The paper presents a device for constructing a mechanical apparatus, approximating the solutions of difference, differential, and mixed difference and differential equations of the linear type with constant coefficients, that is comprised of two horizontal metal disks supported at the center and operated in conjunction with a set of graded weights. *S. Ichimura* (Osaka).

★**Dhen, W. Entwicklungsbericht über die repetierende elektronische Analogie-Rechenanlage Darmstadt.** Aktuelle Probleme der Rechentechnik. Bericht über das Internationale Mathematiker-Kolloquium, Dresden, 22. bis 27. November 1955, pp. 87–92. VEB Deutscher Verlag der Wissenschaften, Berlin, 1957.

See also: Foundations, Theory of Sets, Logic: Markov. **Elasticity, Plasticity:** Klein. **Economics, Management Science:** Amsler.

PROBABILITY

Nasr, Saad K. Des positions typiques d'une variable aléatoire. Publ. Inst. Statist. Univ. Paris **5** (1956), 33–42.

Let X be a random variable taking values in a separated uniform space $\mathfrak{X}$ with the uniform structure defined by a family of pseudo-metrics d_i $(i \in I)$. The author defines a typical position of X, $\tau(X)$, as a point in $\mathfrak{X}$ satisfying $d_i(a, \tau(X)) \leq T(d_i(a. X))$ for every $i \in I$ and $a \in \mathfrak{X}$, where $T(N)$ is the typical position of a random number N. He proves that every typical position of a random variable in the sense of Fréchet [Ann. Sci. Ecole Norm. Sup. (3) **65** (1948), 211–237; MR **10**, 386] is also a typical position defined above, and the mean of a Banach space valued random variable of Mourier [C. R. Acad. Sci. Paris **229** (1949), 1300–1301, MR **11**, 376] and the mean of a metric space valued random variable of Doss [Bull. Sci. Math. (2) **73** (1949), 48–72; MR **11**, 190] are also typical positions in his sense. *S. C. Moy* (Detroit, Mich.).

Fréchet, Maurice. Sur diverses définitions de la moyenne d'un élément aléatoire de nature quelconque. Giorn. Ist. Ital. Attuari **19** (1956), 1–15.

Expository paper. *J. L. Doob* (Urbana, Ill.).

Zinger, A. A.; and Linnik, Yu. V. On a certain theorem of the theory of differential equations and "invariant in the mean" statistics. Dokl. Akad. Nauk SSSR (N.S.) **108** (1956), 577–579. (Russian)

Let $P(x_1, \cdots, x_n)$ be a polynomial in the random variables $x_1, \cdots, x_n$, which have independent one-dimensional distributions. One terms P invariant in the mean with respect to $T(x_1, \cdots, x_n)$ if the conditional expectation of P for fixed T is independent of the value of T. The author states sufficient conditions that, when such a polynomial is invariant in the mean with respect to $T=x_1+\cdots+x_n$, one can conclude that the one-dimensional distributions are normal. He shows how this theorem can be deduced from a theorem on analyticity of solutions of an algebraic differential equation.

W. Kaplan (Ann Arbor, Mich.).

Régnier, André. Sur les perturbations aléatoires des systèmes d'équations différentielles. C. R. Acad. Sci. Paris **244** (1957), 2465–2467.

The author considers a differential system $dx_t/dt = U(x_t, a_t)$ where x_t and a_t are vectors in their respective finite dimensional euclidean spaces for each t and where a probability distribution is assigned over a space each point of which is a pair consisting of a function a and an initial condition x_0 associated with the system. He studies certain invariant properties of the resulting a priori distribution of x_t. *A. H. Copeland, Sr.*

Gulotta, Beniamino. Leggi di probabilità condizionatamente stabili. Giorn. Ist. Ital. Attuari **19** (1956), 22–30.

A distribution law L is defined to be conditionally stable with respect to the sequence of joint distribution $L_1, L_2, \cdots$ if, for each n, every non-negative linear combination of n random variables with joint distribution L_n has a distribution of type L. The author gives an example of a law which is conditionally stable but not stable in the usual sense.

A. H. Copeland, Sr.

Császár, Á. Sur une caractérisation de la répartition normale de probabilités. Acta Math. Acad. Sci. Hungar. **7** (1956), 359–382. (Russian summary)

In a normal distribution, the probability of a set of deviations from the mean is greater than the probability of the same deviations from any other value. Conversely, if a distribution has this property, it is normal, given suitable continuity properties. The present paper weakens the hypothesis on these properties, especially by removing an assumption of differentiability. *A. Blake.*

Mallows, C. L. Generalizations of Tchebycheff's inequalities. J. Roy. Statist. Soc. Ser. B. **18** (1956), 139–168; discussion 168–176.

The author defines the class $S(k, \lambda)$ of cumulative distribution functions $F(x)$ satisfying the "smoothness condition (k, λ)" as consisting of all $F(x)$ such that a) the $(k+1)$-st derivative exists and is continuous for all x such that $0<F(x)<1$, and b) there exist constants $\beta_0, \beta_1, \cdots, \beta_{k+1}$ such that $0<(-1)^i F^{(k+1)}(x)<\lambda$ for $\beta_i<x<\beta_{i+1}$ $(i=0, 1, \cdots, k)$, where $F(\beta_0)=0$, $F(\beta_{k+1})=1$. (Hence possibly $\beta_0=-\infty$ or $\beta_{k+1}=+\infty$). He then states the problem: for given $k, \lambda, \mu_1, \cdots, \mu_m$, determine the best possible functions $L(x)$, $U(x)$ such that if 1) $\int_{-\infty}^{+\infty} x^s\, dF(x)=\mu_s$ $(s=1, \cdots, m)$ and 2) $F(x) \in S(k, \lambda)$, then $L(x)<F(x)<U(x)$ for all x such that $0<F(x)<1$. For $k=0$, $\lambda=\infty$ this problem is answered by Chebyshev's inequality. The author then presents a theory leading to interesting results, too numerous and detailed to be reproduced here, which either are solutions of special cases of the problem or lend support to some general conjectures.

Z. W. Birnbaum (Seattle, Wash.).

Nicholson, W. L. On the normal approximation to the hypergeometric distribution. Ann. Math. Statist. **27** (1956), 471–483.

The paper gives an approximation to a sum of hypergeometric terms, and also upper and lower bounds for the sum are given. *A. Jensen* (Lyngby).

Wintner, Aurel. Des distributions symétriques à fonctions caractéristiques convexes. Publ. Inst. Statist. Univ. Paris **5** (1956), 43–46.

Let ϕ be even and convex on $(0, \infty)$, with $\phi(0)=1$, $\phi(\infty)=0$, so that ϕ is the characteristic function of a probability density f. Then f is continuous at 0 if and only if $\phi \in L$, and indeed $f(0+)=\infty$ if $\int^{\infty} \phi$ diverges.

R. P. Boas, Jr. (Evanston, Ill.).

Girault, Maurice. Les fonctions caractéristiques et leurs transformations. Publ. Inst. Statist. Univ. Paris **4** (1955), 223–299.

This booklet presents a synthesis of properties of characteristic functions emphasizing those which characterize three types of distributions: completely discontinuous, absolutely continuous and singular. The following are the main topics: the transformation H, $H\varphi(t)=t^{-1}\int_0^t \varphi(t)dt$ and unimodal distributions; the composition φ_3 of two characteristic functions φ_1, φ_2,

$$\varphi_3(t)=\lim_{u\to\infty}(2u)^{-1}\int_{-u}^{u}\varphi_1(t+u)\bar{\varphi}_2(u)du;$$

non-negative and positive definite functions; convex laws. The booklet is self-contained and easily readable with few proofs omitted. *S. C. Moy.*

Ibragimov, I. A. On the composition of unimodal distributions. Teor. Veroyatnost. i Primenen. **1** (1956), 283–288. (Russian. English summary)

It is known that the convolution of two unimodal distributions need not be unimodal [cf. K. L. Chung, C. R. Acad. Sci. Paris **236** (1953), 583–584; MR **14**, 771]. The author calls a distribution strongly unimodal when its convolution with any unimodal distribution is unimodal. He shows that if $F(x)$ is the cumulative distribution function of a (non-degenerate) unimodal distribution then a necessary and sufficient condition for this distribution to be strongly unimodal is that $F(x)$ should be continuous and $\log F'(x)$ concave on the set of points at which neither the right-hand nor the left-hand derivative of $F(x)$ vanishes. *H. P. Mulholland* (Birmingham).

★ **Tedeschi, Bruno. Sulle limitazioni più convenienti della probabilità che una variabile casuale a più dimensioni assuma un valore appartenente a un campo assegnato.** Scritti matematici in onore di Filippo Sibirani, pp. 261–279. Cesare Zuffi, Bologna, 1957.

Let $F(a, b)$ be the conjugate distribution function of the two random variables X and Y, C a measurable set of R^2 and $\bar{C}$ its complementary set, $\varphi(x, y; r, s)$ a continuous positive function of x and y, so that $\inf_D (x, y; r, s)>0$ for every point (r, s) in the given domain D, then, as is known,

$$\overline{\varphi(X, Y; r, s)}=Q\inf_{\bar{D}} \varphi(x, y; r, s),$$

where $1-Q=P=\Pr\{X=x, Y=y;\ (x, y) \in C\}$. The author applies this formula for different domains C and obtains limits for the probability P which could be useful.

O. Onicescu (Bucharest).

Urbanik, K. Stochastic processes whose sample functions are distributions. Teor. Veroyatnost. i Primenen. **1** (1956), 146–149. (Russian. English summary)

This paper gives definitions that permit a rigorous discussion of certain stochastic processes used in physics, such as the derivative of the Wiener process. The sequence of stochastic processes $\{f_n(\omega, t)\}$ "converges", $f_n \to f$, if with probability 1 (w.p. 1) there is uniform convergence on every finite t-interval. A sequence $\{f_n\}$ with continuous sample functions is "fundamental" if there is a convergent sequence of processes F_n and integer $k \geq 0$ such that $F_n^{(k)}=f_n$ w.p. 1. Two sequences $\{f_n\}$ and $\{g_n\}$ are "equivalent" if, w.p. 1, $f_n=F_n^{(k)}$, $g_n=G_n^{(k)}$, $F_n-G_n \to 0$. A fundamental sequence determines a "generalized stochastic process", $\Phi=[f_n]$, Φ being the same for all sequences equivalent to $\{f_n\}$. The f_n sample functions define, w.p. 1, a distribution in the sense of Mikusiński [Bull. Acad. Polon. Sci. Cl. III. **3** (1955), 589–591; MR **17**, 594; for connection with the "distributions" of L. Schwartz, see J. Korevaar, Nederl. Akad. Wetensch. Proc. Ser. A. **58** (1955), 663–674; MR **17**, 594]. It is always possible to determine a given Φ by means of a sequence h_n whose sample functions are infinitely often differentiable w. p. 1, and $d\Phi/dt$ is defined as $[h'_n]$. $\Phi=[f_n]$ has independent (increments, values) if for each $\varepsilon>0$ and sufficiently large n, f_n has independent (increments, values) for (intervals, points) whose distance is $\geq \varepsilon$. Theorem: A generalized process has independent values if and only if it is a first order derivative of some generalized process with independent increments. Also treated are correlation functions, spectral representations, and an ergodic theorem for generalized second-order stationary processes. Proofs are not included. *T. E. Harris.*

Akaike, Hirotugu. On a zero-one process and some of its applications. Ann. Inst. Statist. Math., Tokyo **8** (1956), 87–94.

The author considers first a stochastic process $\{x_n(\omega): n=0, 1, 2, \cdots\}$, where $x_0(\omega)=1$ and $x_n(\omega)=1$ or 0, the process being governed by the gap distribution

$$\Pr\{x_{n+1}=x_{n+2}=\cdots=x_{n+\nu-1}=0,\ x_{n+\nu}=1 \mid x_n=1\}=p_\nu$$

($\nu=1, 2, \cdots$). The probability P_n that $x_n(\omega)=1$ can then be calculated from the p_ν's for $n=0, 1, 2, \cdots$ successively, and further (*) $\Pr\{x_n(\omega)=x_{n+\nu_1}(\omega)=\cdots=x_{n+\nu_1+\cdots+\nu_k}(\omega)=1\}=P_nP_{\nu_1}\cdots P_{\nu_k}$ for $k=0, 1, 2, \cdots$. Let the highest common factor of integers ν with $p_\nu>0$ be 1. Then it is known that P_n tends to $P=1/\sum \nu p_\nu$ as $n\to\infty$. This leads the author to consider a second process in which the relation (*) is modified by the substitution of P for P_n (the numbers P_n and P being the same functions of the p_ν's as before). Under the restriction that $p_\nu=0$ for $\nu>N$, say, he proves that the process $\{x_n(\omega)-P\}$ has the continuous spectral density function

$$G'(\lambda)=2R(0)+\sum_{n=1}^{\infty} R(n)\cos 2\pi n\lambda,$$

where $R(n)=(P_n-P)P$. He applies his results to certain observations on automobile traffic and on the winding of silk filaments from cocoons. *H. P. Mulholland.*

★**Hille, Einar. Perturbation methods in the study of Kolmogoroff's equations.** Proceedings of the International Congress of Mathematicians, 1954, Amsterdam, vol. III, p.p. 365–376. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam, 1956. $7.00.

When the number of states in a Markov process is countably infinite, Kolmogoroff's equations can be written in the form

$$-\frac{\partial}{\partial s}Y(s,t)=A(s)Y(s,t)\quad (s\leqq t,\ Y(t,t)=I),$$

$$\frac{\partial}{\partial t}Z(s,t)=Z(s,t)A(t)\quad (s\geqq t,\ Z(s,s)=I),$$

where Y and Z are infinite unknown matrices and where $A(s)=(a_{jk}(s))$ is a given matrix satisfying $a_{jk}(s)\geqq 0$ $(j\neq k)$, $a_{jj}(s)\leqq 0$, $\sum_{k=1}^{\infty} a_{jk}(s)\leqq 0$. Thus in each row of $A(s)$ the diagonal element dominates the others, and it is convenient to write $A(s)=D(s)+T(s)$, where $D(s)$ is a diagonal matrix and $T(s)$ has zero diagonal. When $T(s)=0$, the solution can be written down immediately. In the general case, $T(s)$ may be regarded as a perturbation matrix, and the existence of a solution can be demonstrated by an iterative process which, in fact, leads to a minimal solution $P(s,t)$, common to both equations. The author remarks that his method of proof is essentially the same as that used by W. Feller [Trans. Amer. Math. Soc. **48** (1940), 488–515; MR **2**, 101] for a much wider class of processes, and he acknowledges that several of his arguments were anticipated by J. L. Doob [ibid. **58** (1945), 455–473; MR **7**, 210].

The uniqueness of the solution is discussed only in the temporally homogeneous case, when $A(s)$ is a constant matrix. The "backward" equation then takes the form $Y'(t)=AY(t)$, and the problem is to investigate the existence of null solutions, a distinction being made between weak null solutions satisfying $\lim_{t\to 0+} y_{jk}(t)=0$ and strong null solutions, for which $\sup_j \sum_{k=1}^{\infty} |y_{jk}(t)|\to 0$, as $t\to 0+$. Necessary and sufficient conditions are obtained for the existence of strong null solutions satisfying certain restrictions as to their growth with t. A further application of perturbation methods is made in order to establish the existence of null solutions of $Q'(t)=Q(t)(U+V)$ given that there are null solutions of $Z'(t)=Z(t)U$.

W. Ledermann (Manchester).

Feller, William; and McKean, Henry P., Jr. A diffusion equivalent to a countable Markov chain. Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 351–354.

Let $C[0, 1]$ be the space of real-valued continuous functions on $[0, 1]$, and let the measure $m(dx)$ be positive only on rational numbers $\{r_i\}$ in $(0, 1)$. Let B be the second order differential operator D_mD_x introduced by W. Feller. Given $v\in C[0, 1]$, write $v\in D(B)$, if $Bv\in C[0, 1]$ and if v satisfies the classical boundary condition $v^+(0)=0=v^-(0)$. The operator B contracted to the domain $D(B)$ generates a semi-group T_t $(t>0)$ mapping $C[0, 1]$ into $C[0, 1]$. The Green operator $G_\mu=\int_0^\infty \exp(-\mu t)T_t dt$ $(\mu>0)$ satisfies $(G_\mu v)(r_i)=\sum_{k>0} G(\mu, r_i, r_k)v(r_k)m(r_k)$ and $(T_t v)(r_i)=\sum_{k>0} P(t, r_i, r_k)v(r_k)$, $G(\mu, \cdots, \cdots)$ being the appropriate Green's function of the equation $(\mu-B)w=v$. The proof is based upon the positiveness of the Green's function G. It is also proved that, as $t\downarrow 0$, the denumerable Markoff chain $p(t)=P(t, r_i, r_k)$ satisfies $p(t)\to 1$, $t^{-1}p_{ik}(t)\to 0$ $(i\neq k)$ and $t^{-1}(1-p_{ii}(t))\to+\infty$. *K. Yosida* (Tokyo).

Takashima, Michio. Note on evolutionary processes. Bull. Math. Statist. **7** (1956), 18–24.

Prendiville has pointed out (Jour. Roy. Statist. Soc. B, **11**, 1949, 273) that if the instantaneous birth and death rates in the general birth-and-death process take the {implausible} form

$$\lambda=\alpha\left\{\frac{N_2}{N(t)}-1\right\}\ \text{ and }\ \mu=\beta\left\{1-\frac{N_1}{N(t)}\right\}$$

respectively, where α, β, N_1 and N_2 are constants and $N(t)$ is the population size at time t, the partial differential equation for the probability-generating function of the population size is linear and of first order and is thus soluble generally. The author obtains the solution and discusses its behavior in limiting cases. *H. L. Seal.*

Tsian Tse-pei. On linear extrapolation of a discrete homogeneous stochastic field. Dokl. Akad. Nauk SSSR (N.S.) **112** (1957), 207–210. (Russian)

The author considers prediction theory for a stochastic process $\{x(s, t), -\infty<s, t<\infty\}$, where s, t are integers, the random variables have zero expectations, and there is second order stationarity in the parameter pair, so that the covariance function is given by

$$M\{x(s+m, t+n)\bar{x}(m, n)=\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{i(s\lambda+t\mu)}dF_x(\lambda, \mu).$$

The linear least squares prediction problem, as formulated by the author, is (roughly) that of approximating the infinite dimensional vector $x(\cdot, t)$, for specified positive t, by linear combinations of vectors $x(\cdot, t')$ with $t'\leqq 0$. Let H_x be the closed linear manifold of random variables generated by all the $x(m, n)$, let $H_x(t)$ be like H_x except that $n\leqq t$, and let $S_x=\bigcap_t H_x(t)$. If $S_x=H_x$, the process is called singular. If $\eta(s, t)$ is the projection of $x(s, t)$ on $H_x(0)$, and if the L_2 norm $\|x(s, t)-\eta(s, t)\|$ approaches $\|x(0, 0)\|$ when $t\to\infty$, the process is called regular. The following are among the theorems stated without proof. (1) The process is regular if and only if (a) F (planar) measure and $dF_x(\lambda, \pi)\times d\mu$ product measures are ab-

solutely continuous with respect to each other on the obvious square, and (b)

$$\left| \int_{-\pi}^{\pi} \log \left(\frac{dF_x(\lambda, \mu)}{dF_x(\lambda, \pi) d\mu} \right) d\mu \right| < \infty$$

for almost all $\lambda(dF_x(\lambda, \pi)$ measure). (2) The process is singular if and only if the above integral, with the usual interpretation of the integrand as a Radon Nikodym derivative, has the value $+\infty$. The one step prediction error is evaluated explicitly, and a property like the Markov property is studied. *J. L. Doob* (Urbana).

Longuet-Higgins, M. S. The statistical analysis of a random, moving surface. Philos. Trans. Roy. Soc. London Ser. A. **249** (1957), 321–387.

The physical problem which motivates this investigation of random, moving surface is that of analyzing the motion of waves at sea. The main part of the paper, however, consists of a systematic study of two-dimensional wave motion using generalizations of the principles developed in the study of time series and random noise. The paper is divided into three parts. Part I, "Description of the Surfaces" (10 pages), investigates the simple wave, the superposition of waves and the connection between the shape of a surface and the spectrum of its component waves. Part II, "Statistical Properties" (40 pages), contains an extensive study of various properties starting with the distribution of surface elevation and going through such things as the distribution of zeros, maxima, velocities of maxima, etc. The basic assumption is that the process is the two-dimensional generalization of a stationary Gaussian process. Most of the various properties are related to moments of the two-dimensional spectral density. Part III, "A Method of Determining the Energy Spectrum" (7 pages), treats the problem of finding the spectrum from observations of the heights and slopes of the surface. — The paper is written in a leisurely style but contains a wealth of interesting theory in a well-organized and reasonably self-contained development.

G. Newell (Providence, R.I.).

Khintchine, A. Nachwirkungsfreie Folgen von zufälligen Ereignissen. Teor. Veroyatnost. i Primenen. **1** (1956), 3–18. (Russian. German summary)

The author considers a sequence of random events; the number $x(t)$ of events occurring during the time interval $(0, t)$ is then an integer-valued stochastic process. Suppose that $x(t)$, $(t \geqq 0)$, is a process with independent increments, denote by $\Lambda(t)$ the expectation of $x(t)$ and assume that $\Lambda(t)$ is finite for $t \geqq 0$. A time point t is called regular (respectively, singular) if $\Lambda(t)$ is continuous (respectively, discontinuous) in t. A process which has only regular points is called a regular process and the corresponding sequence of events is called a regular sequence. A sequence of events is called singular if it satisfies the following conditions: (i) the events can occur only at given instants t_i $(i=1, 2, \cdots)$ which form an at most denumerable set, (ii) the numbers of events occurring at different instants t_i are independent random variables, (iii) the expected number of events during the time interval $(0, t)$ is finite for any $t>0$. A singular sequence of events is determined by the sequence of points t_i and by the probabilities q_k^i $(k \geqq 0,\ i \geqq 1)$ that exactly k events occur at the instant t_i.

Consider first a regular process and denote by $v_k(\alpha, \beta) = P\{x(\beta) - x(\alpha) = k\}$ the probability that exactly k events occur during the time interval (α, β) and form the generating function $\Phi(x, \alpha, \beta) = \sum_{k=0}^{\infty} v_k(\alpha, \beta) x^k$. The author characterizes the generating function of regular, integer-valued processes with independent increments and finite mean value function $\Lambda(t)$ and shows that every integer-valued process with independent increments can be decomposed into a regular and a singular component. The parameters of the singular component can be determined from the mean value function $\Lambda(t)$ of the process.

E. Lukacs (Washington, D.C.).

Khintchine, A. Über Poissonsche Folgen zufälliger Ereignisse. Teor. Veroyatnost. i Primenen. **1** (1956), 320–327. (Russian. German summary)

The author continues his earlier investigations [for the notation see the preceding review] on sequences of random events. Denote by $\psi_k(\alpha, \beta) = \sum_{r=k}^{\infty} v_r(\alpha, \beta)$ the probability that at least k events occur in the time interval (α, β). A sequence of events with continuous mean value function $\Lambda(t)$ is called an ordinary sequence if for arbitrary $t>0$, $\varepsilon>0$ there exists a $\delta>0$ such that for $0 \leqq \alpha < \beta < \alpha + \delta < t$ the inequality $\psi_2(\alpha, \beta) \leqq \varepsilon \psi_1(\alpha, \beta)$ holds. The author proves the following theorem. For a regular sequence of events,

$$v_k(\alpha, \beta) = \frac{[\Lambda(\beta) - \Lambda(\alpha)]^k}{k!} \exp\left\{ - \left[\Lambda(\beta) - \Lambda(\alpha) \right] \right\}$$

if, and only if, the sequence is ordinary.

In case the sequence is singular it can be represented as the composition of an ordinary regular sequence and a singular Poissonian sequence. *E. Lukacs.*

Harris, T. E. The random functions of cosmic-ray cascades. Proc. Nat. Acad. Sci. U.S.A. **43** (1957), 509–512.

A photon of energy ε has probability λdt (in the usual, conventional wording) of being transformed in a time interval of length dt into two electrons of energies εu, $\varepsilon(1-u)$, with probability density $q(u)$. An electron loses energy βt in time t, as long as it has strictly positive energy. An electron of energy ε emits, in a time interval of length dt, a photon of energy in the interval $(\varepsilon u, \varepsilon u + \varepsilon du)$ with probability $k(u)du dt$. Here q has a bounded derivative, is symmetric about $\frac{1}{2}$, and $(k(u) - \mu/u)' \leqq c(1-u)^{-b}$, for some positive constants μ, c, $b<2$, so that $\delta = \int_0^1 k(u)du = \infty$. Under these hypotheses, infinitely many photons are produced by an electron in each t interval in which it has positive energy. Generalizing a result of Bethe and Heitler [Proc. Roy. Soc. London. Ser. A. **146** (1934), 83–112], if $\beta=0$, the (infinitely divisible) distribution of $-\log \varepsilon_0(t)$ is found, where $\varepsilon_0(t)$ is the energy at t of an electron of initial energy 1. Let $f_1[f_2]$ be the generating functions of the total number of electrons at t of energies $>\varepsilon$, if initially there is one photon [electron] of energy 1. Differential equations for f_1, f_2 are given. Corresponding results for $\delta < \infty$ have been obtained, for example, by Jánossy [Acta Phys. Acad. Sci. Hungar. **2** (1952), 289–333; MR **15**, 330]. A vector Markov process $\{I(t), \zeta(t), t \geqq 0\}$ is defined, where $I(t)$ has values 1, 2, and $\zeta(t)$ is real-valued. Roughly, $I(t)$ tells whether a single particle is a photon or electron, and $\zeta(t)$ specifies its energy. If $\beta=0$, the probability that $I(t)=j$ and (density) $\zeta(t)=\varepsilon$ when $\zeta(0)=1$, $I(0)=i$ is $\varepsilon m_{ij}(\varepsilon, t)$, where $m_{ij}(\varepsilon, t)$ is the expected number of particles of type j at t of energies between ε and $\varepsilon + d\varepsilon$ starting with one particle of type i and energy 1. Using this device, integral equations can be obtained for the m_{ij}, and a stationary (generalized) distribution for the Markov process yields one for the expected number of photons or electrons in

the energy range $(\varepsilon+d\varepsilon)$. A corresponding result for particles of a single type was given by Jánossy [Proc. Roy. Irish Acad. Sect. A. **53** (1950), 181–188; MR **13**, 569]. If $\beta=0$, let $z(t)$ be the total energy at t in all electrons. It is shown that $\lim_{t\to\infty} z(t)$ is a specified constant, strengthening a result of Bhabha and Chakrabarty [Proc. Roy. Soc. London. Ser. A. **181** (1943), 267–303; MR **4**, 291]. Finally, if $\beta>0$, various results are stated. For example, although the number $N(t)$ of electrons at t of strictly positive energy has a finite second moment, the supremum of $N(t)$ over any interval is ∞.

J. L. Doob (Urbana, Ill.).

Longuet-Higgins, M. S. On the transformation of a continuous spectrum by refraction. Proc. Cambridge Philos. Soc. **53** (1957), 226–229.

If $E(u, v)$ is the spectral density function of a two-dimensional wave train (u, v being wave numbers) then the author considers the transformation of E when the wave train undergoes refraction. If the transformed quantities are distinguished by primes then it is shown that, under certain assumptions, $E'(u', v')=E(u, v)$.

P. Whittle (Wellington).

Longuet-Higgins, M. S. On the velocities of the maxima in a moving wave-form. Proc. Cambridge Philos. Soc. **53** (1957), 230–233.

Given a stationary random function $f(x, t)$ of space coordinate x and time t, it is desired to calculate the probability distribution of the velocity c of points of constant level, $f=\xi$. The author calculates the distribution of c conditional on the three following specifications: (1) that $\xi<f<\xi+d\xi$, (2) that, for a given t, $f=\xi$ somewhere in $(x, x+dx)$, (3) that, for a given x, $f=\xi$ somewhere in $(t, t+dt)$. The calculation yields three distinct distributions supporting the author's contention that the definition of a constant-level point is to some extent arbitrary.

P. Whittle (Wellington).

See also: Foundations, Theory of Sets, Logic: Łuszczewska-Romahnowa; Czerwinski. **Analytic Theory of Numbers:** Postnikov. **Topological Vector Spaces:** Gel'fand. **Statistical Thermodynamics and Mechanics:** Jaynes. **Elasticity, Plasticity:** Eringen. **Optics, Electromagnetic Theory, Circuits:** Bunimovich. **Biology and Sociology:** Rashevsky. **Information and Communication Theory:** Siforov, Jacobsen.

STATISICTS

★**Risser, R.; et Traynard, C.-E. Les principes de la statistique mathématique. Livre I: Séries statistiques.** 2me éd., revue et augmentée. Traité du calcul des probabilités et de ses applications. Tome I, fasc. IV. Gauthier-Villars, Paris, 1957. xvi+195 pp. 3,500 fr.

The present volume (with a second volume covering correlation and regression theory to appear later) is a revision of the first edition which appeared some twenty-five years ago. Unfortunately much of the material of the first edition is retained intact in the first one hundred pages, material either obsolete or of very limited present day interest. In this portion no attempt is made to distinguish population parameters from statistics. Even in the remainder of the text no authorities are quoted since R. A. Fisher. The work of Neyman and Pearson on the power function, that of Wald on statistical decision theory, Monte Carlo techniques, and recent non-parametric methods, are ignored.

The authors cover statistical measures, the usual distribution functions, estimation, sampling, uncertain inference, testing of hypotheses, sampling surveys, the theory of quality control charts, and a sliver of sequential analysis.

L. A. Aroian (Culver City, Calif.).

Adhikari, B. P.; et Joshi, D. D. Distance, discrimination et résumé exhaustif. Publ. Inst. Statist. Univ. Paris **5** (1956), 57–74.

It is suggested that for a distance on a space of probability measures to be useful in statistical applications, it should (i) be a finite metric, (ii) increase with the (random) sample size, (iii) not increase under measurable transformations (and perhaps remain invariant under transformations induced by sufficient statistics), (iv) be a maximum for measures which are singular with respect to one another, (v) be inversely related to the error of discrimination. Distances proposed by Kolmogoroff, Matusita and Chernoff and related to the "affinity" of Bhattacharya are examined. The authors do not rank these, but according to the criteria adduced the prize goes to Japan. Some other distances are also considered.

H. Teicher (Lafayette, Ind.).

Keats, John A. Estimation of error variances of test scores. Psychometrika **22** (1957), 29–41.

Let the responses of a person to a test of n items be considered to be coordinates of a point A in an E_n space, each coordinate having the value 0 or 1. Then the total score may be interpreted as the square of the distance from A to the origin, and points corresponding to equal scores form regular patterns. By assigning equal probabilities to points in patterns for fixed scores, the author derives the distribution of the scores on a half test for persons who have a fixed score on the whole test and then he finds the variance of such half test scores. The formula is valid for test items of equal difficulty; in the case of items of varying difficulty he proposes an adjustment in the formula so that its average value will equal the average error variance over a group. This worked reasonably well in some empirical examples included.

C. C. Craig.

Krishna Iyer, P. V.; and Bhattacharyya, M. N. On some statistics comparing two binomial sequences. J. Indian Soc. Agric. Statist. **7** (1955), 187–217.

Suppose that $x_1, x_2, \cdots x_n$,and $y_1, y_2, \cdots, y_n$ are two binomial sequences in which each x or y is an A or B with probabilities p_1, q_1 in the first sequence and p_2, q_2 in the second. $(p_i+q_i=1;\ i=1, 2)$. The authors study the distribution in samples of three statistics, X_s, Y_s, Z_s as possible test criteria for the null hypothesis that $p_1=p_2=p$. These statistics are defined by:

$$X_s=\sum_{r=1}^{n}(x_r-y_r)+\sum_{i=1}^{s}\sum_{r=1}^{n-i}\{(x_r-y_{r+i})+(x_{r+i}-y_r)\},$$

$$Y_s=\sum_{r=1}^{n}|x_r-y_r|+\sum_{i=1}^{s}\sum_{r=1}^{n-i}\{|x_r-y_{r+i}|+|x_{r+i}-y_r|\},$$

and

$$Z_s=\sum_{r=1}^{n}(x_r-y_r)^2+\sum_{i=1}^{s}\sum_{r=1}^{n-i}\{(x_r-y_{r+i})^2+(x_{r+i}-y_r)^2\},$$

in which $s=0, 1, \cdots, n-1$ in conjunction with the

following scaling system:

$$\begin{aligned} x-y=(A-B)=1 \qquad & |x-y|=|A-B|=1 \\ =(B-A)=-1 \qquad & =|B-A|=1 \\ =(A-A)=0 \qquad & =|A-A|=0 \\ =(B-B)=0 \qquad & =|B-B|=0 \end{aligned}$$

$$\begin{aligned} (x-y)^2=(A-B)^2=1 \\ =(B-A)^2=1 \\ =(A-A)^2=0 \\ =(B-B)^2=0, \end{aligned}$$

This makes Y_s and Z_s identical so that only one need be considered.

For X_0 the probability generating function (PGF) is obtained and the first four cumulants are given. For X_1 and X_2 recurrence formulae for the PGF's are found and the first four cumulants written down. From these results the form of the PGF and hence of the moment generating function is deduced for $s \leqq n/2$ (n even) and for $s \leqq (n+1)/2$ (n odd) and the approximate values of the first four cumulants are given for n large. For the remaining values of s the mean and variance are given. These first two cumulants are given also for $p_1 \neq p_2$.

For Y_0 and Y_1 the like results are found. For Y_2 the first two cumulants are given as is also done for Y_s.

The exact distributions of X_0 for $n=1(1)6$ and of Y_1 for $n=2(1)6$, for $p=0.1(.1).5$, are tabled. It is noted that the cumulants of X_s and Y_s are linear in n and hence it is inferred that these statistics are asymptotically normal. Using this result the power of X_s for testing H_0 ($p_1=p_2=\frac{1}{2}$) against the alternatives $p_2=0.1(.1).4$ for $n=30$, $\alpha=0.05$ and $s=0, 1, 5, 10$ is tabulated. The power of Y_s is also tabulated for the same set of conditions except that here s has the values 0, 1, 5(5) 25, 29. *C. C. Craig.*

Mallows, C. L. Non-null ranking models. I. Biometrika **44** (1957), 114–130.

Moore, P. G. Transformations to normality using fractional powers of the variable. J. Amer. Statist. Assoc. **52** (1957), 237–246.

Let X be a non-negative random variable and define $Z=X^r$ ($0<r<1$). It is desired to choose r so that Z is "approximately" normally distributed. The author gives nomograms to facilitate the choice of r. These are based on fitting the first four moments in the classical method of the Pearsonian school. Examples are given that show that this mathematically vague procedure is reasonably satisfactory from a statistical point of view.

D. G. Chapman (Seattle, Wash.).

Walsh, John E. Estimating population mean, variance, and percentage points from truncated data. Skand. Aktuarietidskr. **39** (1956), 47–58.

Let a sample of size n from a population with continuous cumulative distribution function $F(x)$ be censored so that the n_1 smallest and the n_2 largest values are not available ($0 \leqq n_1+n_2<n$). The assumption is made that, with reasonable approximation, $F(x)$ can be represented in the form

$$\phi[(x-A)/B]+C\phi^{(4)}[(x-A)/B]+\phi^{(6)}[(x-A)/B] \\ +g[(x-A)/B]$$

where ϕ is the standardized normal c.d.f., $\phi^{(j)}$ its j-th derivative, and $g(z)$ an arbitrary continuous even function. A procedure is proposed for obtaining estimates of the 100β per cent point, the standard deviation and the expectation of the population. In a section entitled Statement of Method this procedure is presented step-by-step (eleven steps, some by trial and error) leading either to the required estimates or, possibly, terminating at some of the steps by giving up the attempt to obtain the estimates. In an Appendix the underlying "motivations and mathematical relations" are considered; it is not shown that the proposed estimates have the properties usually expected from "good" estimates. *Z. W. Birnbaum.*

Haldane, J. B. S. Almost unbiassed estimates of functions of frequencies. Sankhyā **17** (1956), 201–208.

"Estimates of powers of a frequency, and of its logarithm, are obtained whose bias falls off more rapidly than any power of the sample number." (From the author's summary.) *L. A. Aroian* (Culver City, Calif.).

Cohen, A. Clifford, Jr. On the solution of estimating equations for truncated and censored samples from normal populations. Biometrika **44** (1957), 225–236.

Let x_0, x_0+w be the smallest and largest values in a sample of n from a $N(\mu, \sigma^2)$ population. The author gives a chart by means of which maximum-likelihood estimates of μ, σ may conveniently be determined from $(\bar{x}-x_0)/w$ and $\sum (x_i-\bar{x})^2/(nw^2)$. Iterative procedures for improving these estimates are discussed. Doubly and singly censored samples are also considered and worked examples given. *H. A. David* (Blacksburg, Va.).

Bliss, C. I. Bioassay from a parabola. Biometrics **13** (1957), 35–50.

The author examines two cases in which the response, y, plotted against the log dose, x, for the standard and each of four unknowns had a slight but significant curvature. The data were fitted by parabolas parallel in y, by lines using a logarithmic transformation of y, and by parabolas parallel in x. The log-relative potencies and the confidence intervals for the unknowns showed relatively small discrepancies among the three methods. In fact, it appears from these examples that if the curvature is small and no fully effective linear transformation of the response is at hand, "the potency computed factorially, as if the response were linear against the log-dose, seems to have so small a bias that the resulting log-potencies and their confidence intervals would classify as 'robust' statistics." *C. C. Craig* (Ann Arbor, Mich.).

Lindley, D. V. Binomial sampling schemes and the concept of information. Biometrika **44** (1957), 179–186.

In random sampling from a binomial distribution it is assumed that θ, the unknown binomial proportion, has a known a priori density, $\theta^{a-1}(1-\theta)^{b-1}\Gamma(a+b)/\{\Gamma(a)\Gamma(b)\}$, with a and b positive. Rules are obtained for sampling until a predetermined amount of information (in the sense of Shannon) is available about: (i) θ; (ii) $2 \arcsin \sqrt{\theta}$; (iii) $\ln\{\theta/(1-\theta)\}$. The relationship between Shannon's concept of information and that of Fisher is discussed, and it provides explanations of some features of the sampling schemes. *W. L. Smith.*

Kitagawa, Tosio. The operational calculus and the estimations of functions of parameter admitting sufficient statistics. Bull. Math. Statist. **6** (1956), 95–108.

The author considers bounded linear translatable

operations Λ_u upon functions $f(u)$ in the Lebesgue class $\mathfrak{L}^p$ on $(-\infty, \infty)$: he defines these as in a previous paper [Jap. J. Math. 22 (1952), 1–18, § 3; MR 16, 151] except that now the functions $f(u)$ are not random functions. He extends his theory to unbounded linear translatable operations Λ_u. Given two functions $\theta(\lambda)$ and $A(\lambda)$ he obtains, under assumptions too numerous for reproduction here, the formula $\Lambda_u[e^{\lambda u}]=\theta(\lambda)e^{\lambda u}$, and, defining $e^{v(u)}$ and $e^{v(u)}W(u)$ by suitable complex integrals involving $\theta(\lambda)$ and $A(\lambda)$, he shows that

$$A(\tau)\int_{-\infty}^{\infty} W(u)\exp\{-\tau u+v(u)\}du=\theta(\tau).$$

Thus $W(u)$ is an unbiased estimate for $\theta(\tau)$ if u is a sufficient statistic for τ whose probability element has the form $A(\tau)\exp\{-\tau u+v(u)\}$. The requisite assumptions are fulfilled, in particular, when the probability element is $(2\pi n)^{-\frac{1}{2}}\exp(\frac{1}{2}n\lambda^2)\exp(-\lambda u-\frac{1}{2}u^2n^{-1})du$. The author notes that his treatment is alternative to that of Y. Washio, H. Morimoto, and N. Ikeda [Bull. Math. Statist. 6 (1956), 69–93; MR 18, 772]. He considers also biased estimates designed to minimize a "loss-function".

H. P. Mulholland (Exeter).

Kramer, Clyde Young. Extension of multiple range tests to group correlated adjusted means. Biometrics 13 (1957), 13–18.

The general problem is to modify D. B. Duncan's "New multiple range test" [Biometrics 11 (1955), 1–42; MR 16, 842] to accommodate adjusted means which are correlated. For k means $\bar{y}_1, \bar{y}_2, \cdots, \bar{y}_k$ satisfying $\bar{y}_1\leqq\bar{y}_2\leqq\cdots\leqq\bar{y}_k$, with estimated variances $c_{ii}s^2$ and covariances $c_{ij}s^2$ $(i<j;\ i, j=1, \cdots, k)$, it is proposed that the difference $\bar{y}_i-\bar{y}_j$ be judged significant if (1) the difference $\bar{y}_{i'}-\bar{y}_{j'}$ is significant for all i', j' such that $i'\leqq i$, $j\leqq j'$, and (2) $\bar{y}_i-\bar{y}_j>z_{p,n_2}[\frac{1}{2}(c_{ii}-2c_{ij}+c_{jj})s^2]^{\frac{1}{2}}$, where z_{p,n_2} is a critical value from the upper tail of the distribution of the studentized range of p treatments, with an error mean square based on n_2 degrees of freedom. Special cases considered are covariance analysis, balanced and partially balanced incomplete block designs, lattices, and the one way classification with unequal frequencies.

W. S. Connor (Washington, D.C.).

★ **Gnedenko, Boris. Über die Nachprüfung statistischer Hypothesen mit Hilfe der Variationsreihe.** Bericht über die Tagung Wahrscheinlichkeitsrechnung und mathematische Statistik in Berlin, Oktober, 1954, pp. 97–107. Deutscher Verlag der Wissenschaften, Berlin, 1956.

This is an expository paper summarizing a wealth of results obtained by the author and by his colleagues at the University of Kiev. These results were originally published either in the Doklady or in the Uspekhi of the Russian Academy and the paper contains a list of 31 references. The subjects reported are connected with the Kolmogorov-Smirnov test and represent distributions of various criteria as determined by the hypothesis tested. One point requires comment. There is a complete lack of any consideration of the power of the tests considered. In fact, even when the author expresses a preference for a particular test criterion compared with another, this is done solely on the basis of the distribution of these criteria as determined by the hypothesis tested. Consider the hypothesis H asserting that an observable r.v. X has a specified distribution function $F(x)$. Also, let $S_n(x)$ denote the empirical distribution function based on n independent observations on X. Finally, let Δ_n be the measure of the projection on the axis of ordinates of the set of all those points $(x, F(x))$ for which $S_n(x)>F(x)$. The author quotes the already published result that, if the hypothesis H be true, then the distribution of Δ_n is uniform between zero and unity. This fact is interpreted as an indication that Δ_n is worthless as a criterion for testing the hypothesis H. In this connection it would be interesting to investigate the power of the obvious test based on Δ_n for the case where the set of admissible hypotheses is limited to those ascribing to X the distribution $F(x-\theta)$, where θ is an arbitrary real number.

J. Neyman.

Sarhan, A. E.; and Greenberg, B. G. Tables for best linear estimates by order statistics of the parameters of single exponential distributions from singly and doubly censored samples. J. Amer. Statist. Assoc. 52 (1957), 58–87.

Tables are given for sample sizes up to ten. Both one and two parameter exponential distributions are considered. Related results may be found in B. Epstein, Ann. Inst. Statist. Math., Tokyo 8 (1956), 15–26 [MR 18, 344].

B. Epstein (Stanford, Calif.).

Savage, I. Richard. On the independence of tests of randomness and other hypotheses. J. Amer. Statist. Assoc. 52 (1957), 53–57.

Siotani, Minoru. Order statistics for discrete case with a numerical application to the binomial distribution. Ann. Inst. Statist. Math., Tokyo 8 (1956), 95–104.

The author finds the distribution of the extremes and the range in samples from a discrete population. He appears unaware of work by Abdel-Aty [Statistica, den Haag 8 (1954), 61–82; MR 16, 729] and Burr [Ann. Math. Statist. 26 (1955), 530–532; MR 17, 278]. *H. A. David.*

Dwass, Meyer. Modified randomization tests for nonparametric hypotheses. Ann. Math. Statist. 28 (1957), 181–187.

Tests of the invariance of the distribution of $z=(z_1, \cdots, z_N)$, which reject if at most $k=\alpha N!$ permutations have $u(pz)\geqq u(z)$, are approximated by tests which reject if at most $d=\alpha(s+1)-1$ of s independent random permutations have $u(P_iz)\geqq u(z)$. To within k^{-1}, the modified test has size α and, over a class of alternatives including all against which the original test is best, has minimum relative power

$$1-(1-\alpha)b(d;\ s,\ \alpha)\geqq 1-[(1-\alpha)/2\pi\alpha s]^{\frac{1}{2}}.$$

J. Hannan (East Lansing, Mich.).

Das, M. N. Latin squares with several missing plots. J. Indian Soc. Agric. Statist. 7 (1955), 46–56.

Formulae are developed for analyzing a Latin Square when (1) the number of values missing is less than the number of treatments and each belongs to a different row, column and treatment, and (2) a row, column or treatment is partly but not completely missing. An example of (1) is worked out. *W. S. Connor* (Washington, D.C.).

Cornfield, Jerome; and Tukey, John W. Average values of mean squares in factorials. Ann. Math. Statist. 27 (1956), 907–949.

This is an important paper. It is an extension of Tukey's now classical research report [Statist. Res. Group, Princeton Univ., Memo. Rep. 18 (1949)], although the

derivations in the present paper more closely follow an independent research report by Cornfield. The paper deals with a very general linear model, which covers crossed, nested, and, for the case without interactions with experimental units, completely randomized classifications. Simple rules are derived, by means of which formulas for average mean squares in analysis of variance can be easily constructed for replicated multifactorial experiments, including both fixed and random factors and their interactions. The model is very flexible, since all populations, including those corresponding to errors, are assumed finite. Explicit formulas are given for two-way and three-way classifications. The relation of this paper to other work, published and unpublished, is discussed.

D. M. Sandelius (Göteborg).

Wilk, M. B.; and Kempthorne, O. Some aspects of the analysis of factorial experiments in a completely randomized design. Ann. Math. Statist. **27** (1956), 950–985.

Consider a completely randomized factorial design with factors $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ such that the number of levels of each factor are A, B, and C respectively. Let the factors be (arbitrarily) ordered and let $i=1, 2, \cdots, A$; $j=1, 2, \cdots, B$; $k=1, 2, \cdots, C$ denote the various levels of factors $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$. Suppose that there are P experimental units available to study the various treatment combinations and let $m=1, 2, \cdots, P$ denote the unit in the population of units. The experimental design is defined by (i) select a, b, and c levels at random from factors $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$, (ii) select p experimental units at random from P. The notation is such that $i^*=1, 2, \cdots, a$; $j^*=1, 2, \cdots, b$; $k^*=1, 2, \cdots, c$ denote the randomly selected levels; $(i^*j^*k^*)$ denotes a randomly chosen treatment combination; and $n_{i^*j^*k^*} \geqq 1$ are pre-chosen fixed numbers such that $n_{i^*j^*k^*}$ is the number of units on which treatment $(i^*j^*k^*)$ appears. This is a general description of a linear model which includes as special cases (1) fixed model $(a=A, b=B, c=C)$, (2) random model $(a \ll A, b \ll B, c \ll C)$ and (3) mixed model (combination of (1) and (2)).

Let Y_{ijkm} represent the "true" response if unit m is subjected to treatment combination (ijk). However, the quantity actually observed in y_{ijkm} where $y_{ijkm}=Y_{ijkm}+\varepsilon_{ijkm}$ and the ε_{ijkm} are such that $E(\varepsilon_{ijkm})=0$, $E(\varepsilon_{ijkm}^2)=\sigma^2$. The authors define a population model by

$$y_{ijkm}=\mu+a_i+b_j+c_k+(ab)_{ij}+(ac)_{ik}+(bc)_{jk}+(abc)_{ijk}+p_m+q_{ijkm}+\varepsilon_{ijkm},$$

where (using a dot notation to indicate an average over a particular index)

$$\mu=Y_{\cdots},\ a_i=Y_{i\cdots}-\mu,\ \cdots,$$

$$(ab)_{ij}=Y_{ij\cdot\cdot}-Y_{i\cdots}-Y_{\cdot j\cdot\cdot}+\mu,\ \cdots,$$

$$(abc)_{ijk}=Y_{ijk\cdot}-Y_{ij\cdot\cdot}-Y_{i\cdot k\cdot}+Y_{i\cdots}-(bc)_{jk},$$

$$p_m=Y_{\cdots m}-\mu,\ q_{ijkm}=Y_{ijkm}-Y_{ijk\cdot}-Y_{\cdots m}+\mu.$$

The quantity p_m is called a unit error, and q_{ijkm} represents the interaction of treatment combination $(ijkm)$ with unit m.

Using this general set-up, the authors derive the expected values of the various mean squares associated with the analysis of variance for a completely randomized design. They consider (i) the case of proportional numbers, i.e., $n_{i^*j^*k^*}=rw_{i^*}v_{j^*}w_{k^*}$, (ii) an orthogonal analysis based on cell averages, (iii) a non-orthogonal analysis using unweighted cell means. The results on the various expected values of mean squares are too complicated for this review. However, the general results are that unbiased estimates of the variance components (say) $\sigma_a^2=(A-1)^{-1}\sum_{i=1}^{A} a_i$, etc. cannot be obtained from the analysis of variance mean squares if unit-treatment interactions are not negligible. However, they state that in many situations this bias will not be important. [Related work on linear models for the case where P is infinite has been done in the paper reviewed above.] *M. Zelen.*

★ **Clatworthy, Willard H. Contributions on partially balanced incomplete block designs with two associate classes.** National Bureau of Standards, Applied Mathematics Series, no. 47. U. S. Government Printing Office, Washington, D. C., 1956. iv+70 pp. 45 cents.

The booklet is divided into seven parts with headings as follows: I. On the enumeration of partially balanced designs with two associate classes. II. Some new symmetrical group divisible designs. III. On the duals of partially balanced designs. IV. Triangular incomplete block designs. V. Some Latin square type deisgns with two constraints. VI. Tables of partially balanced designs with two associates classe. VII. References.

Each of the first five parts contains a brief theoretical discussion, an enumeration of parameters of designs of a size not too large for experimental use, and examples either showing how to construct designs or proving nonexistence. Part IV considers 76 known triangular designs. Part VI (which occupies 41 of the 70 pages) lists the parameters for about 400 designs, including roughly 75 new ones. Source references are given for each of the known designs and a construction is given for each new one. Part VII consists of 18 references to the literature.

R. H. Bruck (Madison, Wis.).

Das, M. N. Missing plots in partially balanced and other incomplete block designs. J. Indian Soc. Agric. Statist. **7** (1955), 111–126.

Formulae are developed for analyzing balanced and partially balanced incomplete block designs, Youden squares, and square lattices with m replications. For the case of two missing values, estimates of the missing values, and the variances of the differences between estimated treatment effects are derived. An example of a partially balanced design with two missing values is worked out.

W. S. Connor (Washington, D.C.).

Anderson, T. W. Maximum likelihood estimates for a multivariate normal distribution when some observations are missing. J. Amer. Statist. Assoc. **52** (1957), 200–203.

The results of G. L. Edgett [same J. **51** (1956), 122–131; MR **17**, 891] are proved and generalized by a simplified method. *H. Wold* (Uppsala).

Bradt, Russell N.; and Karlin, Samuel. On the design and comparison of certain dichotomous experiments. Ann. Math. Statist. **27** (1956), 390–409.

Consider the experiments X and Y having distribution F_i and G_i under H_i, where $i=1, 2$. Let ξ denote the a priori probability that H_1 is true and $R_x(\xi)$ the Bayes risk against ξ when using experiment Z. The authors point out

that the problem of obtaining an optimal design by means of computing and comparing $R_x(\xi)$ anfd $R_y(\xi)$ for all ξ is usually intrinsically complicated (an example is given for the binomial distribution to illustrate the complicated structure). A simpler criterion, in terms of Kullback-Leibler information numbers, is proposed. Use of X is recommended whenever $I_x \geq I_y$ (i.e., $I_x(\xi) \geq I_y(\xi)$ for all ξ), where $I_x(\xi) = \xi I_x(1:2) = (1-\xi) I_x(2:1)$. It is shown that this criterion is a necessary condition for $R_x \leq R_y$ (i.e., $R_x(\xi) \leq R_y(\xi)$ for all ξ). For the case where all distributions involved are normal the Kullback-Leibler numbers provide a sufficient condition for $R_x = R_y$, and a second criterion of being "locally more informative" (Bayes) at both 0 and 1 provides a necessary and sufficient condition for $R_x \leq R_y$. *N. C. Severo.*

Bradt, R. N.; Johnson, S. M.; and Karlin, S. On sequential designs for maximizing the sum of n observations. Ann. Math. Statist. **27** (1956), 1060–1074.

For each (p, q) in $[0, 1]^2$ let $(X_1, Y_1), \cdots, (X_n, Y_n)$ be binomial with p.d. $p^{\Sigma x_i}(1-p)^{n-\Sigma x_i} q^{\Sigma y_i}(1-q)^{n-\Sigma y_i}$. A sequential design S is a sequence of functions to $\{0, 1\}$ yielding observed variables $O_k = (s_k(O_1, \cdots, O_{k-1}), Z_k(O_1, \cdots, O_{k-1}))$ with $Z_k = s_k X_k + (1-s_k) Y_k$ $(k = 1, \cdots, n)$. This paper assumes the existence of a known prior distribution F on (p, q) and investigates designs $S^{*n}(F)$ maximizing $E(\Sigma Z_k)$. For 2-point F, a sufficient condition that $S^1(F)$, defined by $s_k^1 = s_k^{*k} (= \text{sign } E[X_k - Y_k | O_1, \cdots, O_{k-1}])$, $CS^{*n}(F)$ is obtained without the restriction to binomials [cf. the paper reviewed above]. Among other results S^{*n} and S^1 are compared as to possession of certain heuristically plausible properties, mostly for F concentrating on $p+q=1$ or for product F. S^{*n} is obtained explicitly for F concentrating on $q=c$ and is identified with the solution of a lot inspection problem. *J. Hannan.*

Bhattacharyya, P. K. Comparison of the means of k normal populations. Calcutta Statist. Assoc. Bull. **7** (1956), 1–16.

Given $\delta, \sigma > 0$ and a sample of n from each of k universes $N(\mu_j, \sigma)$, $j = 1, 2, \cdots, k$, define $R(1) \equiv (\delta \leq \mu_j < \infty)$, $R(2) \equiv (-\delta < \mu_j < \delta)$, $R(3) \equiv (-\infty < \mu_j \leq -\delta)$. Let the decision that $\mu_j \in R(l)$, $l = 1, 2, 3$, be made on the basis of the sample mean $\bar{x}_j$, namely $\mu_j \in R(1)$ if $\bar{x}_j \geq \lambda$, $\mu_j \in R(2)$ if $|\bar{x}_j| < \lambda$ and $\mu_j \in R(3)$ if $\bar{x}_j \leq \lambda$, where λ is the solution of

$$2Q\left(\frac{\lambda-\delta}{\sigma/\sqrt{n}}\right) = Q\left(\frac{-\lambda-\delta}{\sigma/\sqrt{n}}\right), \quad Q(x) = \int_x^\infty (2\pi)^{-\frac{1}{2}} e^{-u^2/2} du.$$

It is shown that this procedure is an admissible minimax decision function in the sense of Wald [Statistical decision functions, Wiley, New York, 1950; MR **12**, 193]. *H. L. Seal* (New York, N.Y.).

Kiefer, J.; and Weiss, Lionel. Some properties of generalized sequential probability ratio tests. Ann. Math. Statist. **28** (1957), 57–74.

Generalized sequential probability ratio tests (GSPRT's for testing between two hypotheses were defined by L. Weiss [same Ann. **24** (1953), 273–281; MR **14**, 889]. The present paper is devoted to a decision-theoretic study of these tests. It is shown that, under suitable regularity conditions, the distributions of the sample size under the two hypotheses uniquely determine a GSPRT, but that, notwithstanding this, many GSPRT's are inadmissible, admissibility being defined in terms of the probabilities of the two types of error and the distributions of the sample size required to come to a decision. It is also shown that, under certain monotonicity assumptions on the probability ratios, the GRSPT's form a complete class with respect to the probabilities of the two types of error and the average distribution of the sample size over a finite set of other distributions. *A. Dvoretzky.*

Hannan, E. J. Testing for serial correlation in least squares regression. Biometrika **44** (1957), 57–66.

Results of Grenander and Rosenblatt [Proc. Nat. Acad. Sci. U.S.A. **40** (1954), 812–816; MR **15**, 974] and Durbin and Watson [Biometrika **37** (1950), 409–428; MR **12**, 512] are used to obtain the asymptotic distribution of a statistic r proposed for use in testing for serial correlation in the residuals of a regression of the form $Y = X\beta + \varepsilon$, with the components of ε generated by a stationary process. Approximate lower bounds for percent points of the distribution of r are found when the regression is on orthogonal polynomials or on orthogonal polynomials and random variables. *F. C. Andrews* (Eugene, Ore.).

Swineford, Frances; and Fan, Chung-Teh. A method of score conversion through item statistics. Psychometrika **22** (1957), 185–188.

Laurent, Andre G. Bombing problems — a statistical approach. Operations Res. **5** (1957), 75–89.

"Bombing problems are studied from the viewpoint of statistical inference and from the viewpoints of the bomber and its opponent. The probability of a hit inside a circle centered at the target is given. A method is proposed to obtain the "best" estimate of the probability of a hit in a given area, with a known or unknown target, on the basis of the information given by the points of impact during a flight. The method is applied to the cases of a rectangle and of a circle centered at the (known) target. Methods of approximation for any area are given. The problem of 'dangerous' and 'safe' regions is discussed. A method to obtain 'distribution free' safe regions is proposed and conditions to obtain 'safe' regions with a given confidence level are stated. Two examples of 'safe confidence regions' are given." (From the author's summary.) *L. A. Aroian* (Culver City, Calif.).

Kitagawa, Tosio; and Seguchi, Tsunetami. The combined use of runs in statistical quality controls. Bull. Math. Statist. **7** (1956), 25–45.

In quality control applied to a continuous variate x, one may use several critical levels, and consider as significant the union of events like "x falls in the outermost zone", "a run of two x's in the second zone", etc. Provided that observation starts afresh after each such instance, the union is again a recurrent event. The authors give formulas and tables for the mean recurrence time under various control schemes of the kind mentioned. *G. Elfving* (Helsingfors).

Fan, Chung-Teh. On the applications of the method of absolute scaling. Psychometrika **22** (1957), 175–183.

See also: Calculus of Variations: Brunk, Ewing and Utz. **Probability:** Longuet-Higgins. **Statistics:** Adhikari and Joshi. **Fluid Mechanics, Acoustics:** Eggleston and Diederich. **Economics, Management Science:** Bonferroni. **Biology and Sociology:** Urbanik.

PHYSICAL APPLICATIONS

Mechanics of Particles and Systems

Rosenauer, N. Synthesis of drag-link mechanisms for producing non-uniform rotational motion within prescribed reduction ratio limits. Austral. J. Appl. Sci. **8** (1957), 1–6.

A drag-link mechanism is a four-bar linkage in which the shortest linkage is fixed and both cranks move in the same direction. The author asks for a linkage in which one crank is rotating uniformly and the extreme values of the output angular velocity are prescribed. A solution is given by means of complex numbers and a numerical example added. *O. Bottema* (Delft).

Fogagnolo Massaglia, Bruna. Sul comportamento di un sistema dissipativo soggetto ad azioni ereditarie non lineari. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **90** (1955–56), 462–471.

The linear dissipative system of one degree of freedom represented by the equation

$$\ddot{q}+2h\varepsilon\dot{q}+\sigma^2 q=0$$

is modified by the addition to the right hand member of a "hereditary" force of the form

$$\varepsilon f(q)-\varepsilon\int_0^{T_0}H(t)\cdot f(q(t-\tau))d\tau.$$

Here h, σ, and T_0 are positive constants, ε is a small positive parameter, and H and f are suitable functions of the indicated variables. The author confines attention to an approximate solution found by the method of Kryloff as applied in non-linear mechanics. It is found that this approximate solution continues to be damped if f is linear; but if f is non-linear, the system may have periodic solutions, either stable or unstable. There is no critical comment on the validity, for the ends in view, of the Kryloff approximation. *D. C. Lewis, Jr.*

Fichera, Elio. Sull'equazione del moto del polo istantaneo di rotazione rispetto al polo medio, in funzione delle coordinate del polo d'inerzia. Rend. Accad. Sci. Fis. Mat. Napoli (4) **23** (1956), 33–48 (1957).

The equations of motion of the instantaneous pole of rotation with respect to the mean north pole of the Earth, which is the origin of the coordinate system (x, y), are given in terms of the coordinates (ξ, η) of the pole of inertia and of the angular velocity $|\omega|$. The integration of these equations is treated in particular cases assuming that the pole of inertia moves: (i) rectilinearly and uniformly; (ii) with negative acceleration proportional to the velocity; (iii) with negative acceleration proportional to the square of the velocity; (iv) with a circular oscillation in the unit of time, and (v) with an elliptical oscillation with constant angular velocity different from ω.
D. Rašković (Belgrade).

Aumann, Georg. Der Raumschlitten. Z. Angew. Math. Mech. **36** (1956), 433–436. (English, French and Russian summaries)

Der Raumschlitten (eine Erweiterung des Carathéodoryschen Schlittens) ist ein starrer Körper, der sich so bewegt, dass in einem körperfesten Punkt die Geschwindigkeit zu einer vorgegebenen körperfesten Richtung entweder senkrecht (I) oder parallel ist (II). Das System ist nichtholonom. Verf. stellt die Bewegungsgleichungen auf für den kräftefreien Schlitten und beschränkt sich dann auf den kugelsymmetrischen Fall. In (I) gibt es nur aperiodische Bewegungen, in (II) sind auch periodische Geschwindigkeitsabläufe möglich. *O. Bottema.*

Basch, A. Zur geometrischen Bestimmung der Hauptschwingungsrichtungen und Eigenfrequenzen eines Systems von zwei Freiheitsgraden aus dessen Charakteristiken. Z. Angew. Math. Mech. **36** (1956), 270–272.

Another version of Ing.-Arch. **10** (1956), 119–124 [MR **18**, 685]. *D. C. Lewis, Jr.* (Baltimore, Md.).

Kosticyn, V. T. On a graphical solution of the problem of motion of a body of variable mass. Trudy Inst. Mašinoved. **15** (1956), no. 60, 45–51. (Russian)

It is shown how the problem of rotation of bodies with variable mass about a fixed axis can be solved graphically, when the equation of motion is $I\ddot{\varphi}+\dot{I}\dot{\varphi}=M$. The moment of inertia I as a function of the angle of rotation φ, and the angular momentum M as a function of time t are given by graphs. *T. P. Anđelić* (Belgrade).

See also: Ordinary Differential Equations: Braunbek. **Partial Differential Equations:** Gould. **Astronomy:** Sklianski; Steffensen.

Statistical Thermodynamics and Mechanics

Furry, W. H. Isotropic rotational Brownian motion. Phys. Rev. (2) **107** (1957), 7–13.

The Brownian motion of the orientation of any rigid body (sphere, set of rectangular axes, etc.) is calculated for the case of individual random infinitesimal rotations whose probabilities are independent of the direction of the axis of rotation (isotropic case). The calculations are carried out by means of quaternions; the more important formulas are also given in terms of the rotation angle and axis direction. The unit solution, or distribution of orientations arising from a specified initial orientation, is found as a series converging rapidly for not-too-small times. It turns out to be expressible in terms of a theta function, and thus there is also available a series converging rapidly for small times. The use of the unit solution as a propagation function is discussed briefly, and is illustrated by verification of the iteration property of the unit solution. *Author's summary.*

Leruste, Philippe. Introduction possible à une mécanique statistique du point aléatoire relativiste. C. R. Acad Sci. Paris **243** (1956), 1102–1105.

Classical statistical mechanics makes use of a 6-dimensional phase-space, to represent the state of a material point; the author suggests an 8-dimensional phase-space in relativistic statistical mechanics: 4 time-space coordinates $x_0=ct$, x_1, x_2, x_3 and 4 impulsion conjugate coordinates p_0, p_1, p_2, p_3. Putting $d\tau^2=dx_0^2-dx_1^2-dx_2^2-dx_3^2$ he considers the 8-vector $U_j=dx_j/d\tau$, $U_{j+4}=dp_j/d\tau$, $j=1, 2, 3, 4$. The generalisation of Liouville's theorem is given in the following form: the flux of this vector is invariant. *J. Kampé de Fériet* (Lille).

MacDonald, W. M.; Rosenbluth, M. N.; and Chuck, Wong. Relaxation of a system of particles with Coulomb interactions. Phys. Rev. (2) **107** (1957), 350–353.

The relaxation to a Maxwellian distribution of a system of particles interacting through inverse-square-law forces is investigated in the approximation of two-particle interactions resulting in small-angle deflections of particle trajectories. The time required for the relaxation of the distribution in the neighborhood of the average energy is found to agree with the self-collision time defined by Spitzer. The time required for the distribution to become Maxwellian throughout the range from zero energy to several times the average energy is found to be nearly ten times the self-collision time. Filling of the high-energy portion of the Maxwell distribution is also discussed.

Authors' summary.

Jaynes, E. T. Information theory and statistical mechanics. Phys. Rev. (2) **106** (1957), 620–630.

The ensemble theory of equilibrium statistical mechanics is derived from the following general principle: the probability distribution p_n over energy levels is such as to maximize the entropy $-\sum p_n \log p_n$ under the restriction that the energy and possibly other quantities have prescribed average values. As is of course well known the conventional equations are immediately obtained in this way. The point of the author is that the above expression of the entropy can now be justified a priori by its significance for information theory, where it was shown by Shannon to be the only expression of uncertainty satisfying a few natural requirements. The author briefly states the reason of the success of ensemble theory for actual physical systems; it is the fact that for many-particle systems the probability distribution of the quantities of interest is very sharply peaked. A few simple examples are considered. *L. Van Hove* (Utrecht).

Elcock, E. W.; and Landsberg, P. T. Temperature dependent energy levels in statistical mechanics. Proc. Phys. Soc. Sect. B. **70** (1957), 161–168.

A general discussion is given of the use of temperature dependent energy levels in statistical mechanics. Such levels can be formally introduced as a result of averages over certain degrees of freedom of the system. The paper analyzes this procedure in general and discusses its limitations. Simple examples are given for illustration.

L. Van Hove (Utrecht).

Rosenbluth, Marshall N.; MacDonald, William M.; and Judd, David L. Fokker-Planck equation for an inverse-square force. Phys. Rev. (2) **107** (1957), 1–6.

The contribution to the Fokker-Planck equation for the distribution function for gases, due to particle-particle interactions in which the fundamental two-body force obeys an inverse square law, is investigated. The coefficients in the equation, $\langle \Delta v \rangle$ (the average change in velocity in a short time) and $\langle \Delta v \Delta v \rangle$, are obtained in terms of two fundamental integrals which are dependent on the distribution function itself. The transformation of the equation to polar coordinates in a case of axial symmetry is carried out. By expanding the distribution function in Legendre functions of the angle, the equation is cast into the form of an infinite set of one-dimensional coupled nonlinear integro-differential equations. If the distribution function is approximated by a finite series, the resultant Fokker-Planck equations may be treated numerically using a computing machine. Keeping only one or two terms in the series corresponds to the approximations of Chandrasekhar, and Cohen, Spitzer and McRoutly, respectively. *Authors' summary.*

Faĭn, V. M. Velocity distribution of electrons in the presence of a variable electrical field and a constant magnetic field. Ž. Eksper. Teoret. Fiz. **28** (1955), 422–430. (Russian)

Approximate solutions of the Boltzmann equation for electrons undergoing elastic collisions with neutral gas molecules (Lorentzian gas) in the presence of a constant magnetic field and of time-dependent electric fields are obtained. Conditions for the validity of the second approximation are discussed in detail. Electric fields of simple harmonic time dependence and amplitude modulated fields are considered. *E. Gora* (Providence, R.I.).

★ **Lindley, D. V. The estimation of velocity distributions from counts.** Proceedings of the International Congress of Mathematicians, 1954, Amsterdam, vol. III, pp. 427–444. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam, 1956. $7.00.

Consider particles in motion throughout a region large compared with a certain region A in which the particles are observed. Counts $\{n_i\}$ $(i=1, 2, \cdots, k)$ of the number of particles in the region A are made at k equally spaced instants of time t, $t+\tau$, $t+2\tau$, $\cdots$, $t+(k-1)\tau$. From these counts and the dimensions of the region A and the assumptions that the particles move independently of one another, it is desired to estimate the speed of the particles. The probability distribution of the stochastic process $\{n_i\}$ is considered and an explicit expression for its cumulant generating function is obtained. Thus, the probability of occurrence of n_1, n_2, n_3 particles in A at instants t, $t+\tau$, $t+2\tau$ depends on the probabilities $p(v)$ and $p(v, w)$ that a particle inside A at time t will be inside A at time $t+v$ and at times $t+v$ and $t+v+w$, respectively; the required cumulant generating function is shown to be

$$\nu\{p(\tau,\tau)\theta_1\theta_2\theta_3+p(\tau)\theta_1\theta_2+p(\tau)\theta_2\theta_3+p(2\tau)\theta_1\theta_3 +\theta_1+\theta_2+\theta_3\},$$

where $\theta_i=e^{z_i}-1$ and ν is the average number of particles in A. This result is readily generalized to include observations at more instants. The main part of the paper is devoted to a rigorous characterization of $p(t)$. This is given by

$$(*) \qquad p(t)=\int_A dx \int_A dy\, \rho(x) f([y-x]/t, t; x)/t,$$

where x and y denote points of A, $\rho(x)$ is the frequency distribution of a randomly observed particle within A, and $f(u, t; x)$ is the frequency distribution of the average velocity, u, of a particle at x, during a subsequent time interval of length t. From (*) the author shows that in the case of one-dimensional motion

$$-2p'(0)\int_0^a dx\nu(x)=\nu(0)c(0)+\nu(a)c(a),$$

where $c(x)$ is the mean speed of the particles at x and $\nu(x)$ is the density of the particles at x. If, further, the direction and magnitude of the velocity of a particle is independent of its position and these occur in accordance with a prescribed distribution function, the equation for $p(t)$ becomes

$$p(t)=\int_0^{a/t} d\omega(1-\omega t/a)g(\omega),$$

where $g(w)$ is the distribution of the particle's speed. This relation between p and the motion is studied and a general method for estimating the mean speed of particles is described. The standard error in the resulting estimation is deduced. Rothschild's [J. Exper. Biol. **30** (1953), 178–199] application of the foregoing stochastic model to estimating the speed of spermatazoa is reviewed.

S. Chandrasekhar (Williams Bay, Wis.).

Bordoni, Piero Giorgio. Sulla teoria statistica dei solidi considerati come insiemi di oscillatori. Ann. Scuola Norm. Sup. Pisa (3) **10** (1956), 237–251 (1957).

Author calculates systematically the value of the coefficient of Grueneisen, γ, in solids assumed to be polycrystal, homogeneous and isotropic media. This coefficient has fundamental significance in the statistical theory of solids. It is composed of two components, γ longitudinal and γ transverse, referred to the longitudinal and transverse types of vibrations. At high temperatures the mean value γ of is expressed by means of a simple formula (sum of one third γ of longitudinal and two thirds of γ transverse). The equation of state depends indirectly on both components of γ, but in the domain of high temperatures it preserves its usual form if the mean value of γ is used. On this basis γ can be calculated from the equation of state using both the statistical expression for the variation of the Poisson coefficient and another coefficient, β, obtainable experimentally. In the next step Bordoni calculates the partial derivatives of the mean value of γ with respect to temperature and specific volume, the statistical expressions for the coefficient of the cubic dilatation and for the coefficient γ as a function of β.

M. Z. v. Krzywoblocki (Urbana, Ill.).

★**Годнев, И. Н. [Godnev, I. N.] Вычисление термодинамических функций по молекулярным данным. [Computation of thermodynamic functions using molecular data.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1956. 419 pp. 13.45 rubles.

The present rapid progress of technology demands a steady supply of numerical data on the thermodynamic properties of various substances. These can be obtained either experimentally or by calculation, from suitably established models, on the basis of statistical and quantum mechanics. The present volume attempts to collect in one place the most important Russian developments on this subject. Some references to Western work are included in the voluminous Bibliography containing 268 titles. References to Chapman and Cowling, to the work done by J. Hirschfelder and his associates and to the recent Nat. Bur. Standards "Tables of thermodynamic properties of gases" (published in 1956 but available in several reports since 1952) are conspicuously absent.

The subject matter covered by the book can be inferred from the titles of its fourteen chapters: 1. Fundamental equations of statistical thermodynamics. 2. Application of the fundamental equations of statistical thermodynamics to perfect gases. 3. Thermodynamic functions for monatomic gases. 4. Equations for the energy and state of diatomic gases. 5. Calculation of the thermodynamic properties of diatomic gases. 6. Classification and types of symmetry in polyatomic gases. 7. Equations for the rotational and vibrational energy of quasi-rigid molecules. 8. Calculation of the thermodynamic properties of (ensembles of) quasi-rigid molecules. 9. The thermodynamic properties of gases for the case of free rotation. 10. Calculation of the thermodynamic properties of gases for the case of inhibited internal rotation. 11. Calculation of the thermodynamic properties of real gases, liquids, and solids with the aid of the corresponding functions in the perfect-gas state. 12. Calculation of chemical equilibria. 13. The calculation of the equilibrium constants for reactions involving isotopes. 14. Tables.

The chapter on real gases (Chap. 11) is entirely based on the use of the Berthelot equation together with the classical equations of thermodynamics and ignores the existence of methods of evaluating virial coefficients by statistical considerations. The very important field of the calculation of transport properties is left unmentioned.

The book is well written and succeeds in presenting its material in a simple, vivid manner, but the reader is left in the dark as to the quality of the agreement with measured data which may be expected from the calculations described. Admittedly, a systematic comparison between the various statistical approximations and existing experimental results, even for gases alone, would present formidable difficulties, but it is a task that is long overdue.

J. Kestin (Providence, R.I.).

See also: Geometries, Euclidean and other: Coxeter. **Fluid Mechanics, Acoustics:** Lighthill. **Quantum Mechanics:** Bocchieri and Loinger; Golden; Gilvarry. **Astronomy:** Piddington.

Elasticity, Plasticity

Aržanyh, I. S.; and Bondarenko, B. A. On the representation of general solutions of the theory of elasticity by definite integrals. Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. **16** (1955), 34–38. (Russian)

Dlugač, M. I. The method of forces as applied to the theory of elasticity. Akad. Nauk Ukrain. RSR. Prikl. Meh. **1** (1955), 83–97. (Ukrainian. Russian summary)

Galimov, K. Z. On variational principles of the nonlinear theory of elasticity. Kazan. Gos. Univ. Uč. Zap. **113**, no. 10 (1953), 155–160. (Russian)

★**Grammel, R., Editor. Verformung und Fliessen des Festkörpers.** Internationale Union für Theoretische und Angewandte Mechanik, Kolloquium Madrid 26. bis 30. September 1955. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1956. xii+324 pp. DM 37.50.

The book contains the papers and edited discussions presented at the 1955 Madrid Colloquium of the International Union of Theoretical and Applied Mechanics. This meeting was an attempt to bring together research workers in applied mechanics and solid state physics concerned with phenomena of deformation and flow of solids. As in most international conferences the papers presented are of uneven quality. There are, however, in this volume a sufficiently large number of first rate papers to make the published proceedings of the conference an outstanding contribution not to be missed by anyone working in this field.

The following is a list of papers presented: Strains in crystalline aggregates (G. I. Taylor); Influence des joints intergranulaires sur l'écrouissage de métaux (B. Jaoul); Versetzungen und Gittertheorie (G. Leibfried); Dis-

locations in crystals (A. H. Cottrell); Theories of fracture in metals (N. F. Mott); Dislocations and brittle fracture in metals (J. R. Low); The origin of dislocations in crystals grown from the melt (F. C. Frank); Magnetoresistivity of plastically deformed metals (H. G. von Buren); Neue mathematische Methoden and physikalische Ergebnisse zur Kristallplastizität (A. Seeger); Les mécanismes de rupture des métaux (C. Crussard, J. Plateau and Y. Morillon); Wave propagation in anelastic materials (E. H. Lee); La structure feuilletée des joints dans les métaux (F. Teissier du Cros); The theory of piecewise linear isotropic plasticity (P. G. Hodge); Nueva teoria de la plasticidad (M. Velasco de Pando); Theory of deformation of non-hardening rigid-plastic plates (H. G. Hopkins); Theory of melting and yield strength (H. Aroeste); Ein nicht-lineares Elastizitätsgesetz (H. Kauderer); Second order effects in infinitesimal elasticity (M. Reiner); Large elastic deformations in rubberlike materials (L. R. G. Treloar); On physical effects in cavitation damage (M. S. Plesset); Methods of "Internal constraints" in dynamic problems (E. Volterra); Variational and Lagrangian methods in viscoelasticity (M. A. Biot); Viscosité et déformations irreversibles (H. LeBoiteux); Viskosität und Zeitwirkungen im nichtlinearen Bereich (F. Schultz-Grunow); Beschädigungs-Prozess in Metallen beim Kriechen (I. A. Oding and W. W. Burdukski); Effect of small viscous inclusions on the mechanical properties of an elastic solid (J. G. Oldroyd); Dislocation relaxation in metals and single crystal quartz (W. P. Mason).

Of these 27 papers about one-half are of at least some mathematical interest. The other half are experimental or purely speculative papers on various aspects of solid state physics, most of which have been published elsewhere in more detail. Of the mathematical papers those by Biot and Hodge represent important contributions to the continuum mechanical theories of visco-elasticity and plasticity respectively; the papers by Lee, Oldroyd and Reiner are interesting illustrations of particular elastic and visco-elastic problems, while de Pando attempts to establish a theory of plasticity on the basis of "hypo-elastic" relations. The paper by Sir Geoffrey Taylor, although not more than a review of his own as well as of other, more recent, work on stress-strain relations in the crystal aggregate, is a little masterpiece of lucidity and simplicity of presentation. With respect to the purpose of the conference the attempt by Seeger to correlate the dislocation theory of slip with the continuum theory of plasticity is of particular interest. *A. M. Freudenthal.*

Paria, Gunadhar. A mixed boundary-value problem of elasticity with parabolic boundary. J. Appl. Mech. 24 (1957), 122–124.

The problem of finding the stress distribution in a two-dimensional elastic body with parabolic boundary, subject to mixed boundary conditions, has been reduced to the solution of the nonhomogeneous Hilbert problem following the method of complex variable. The result has been compared with that for a straight boundary.

Author's summary.

Sengupta, A. M. Some problems of elastic plates containing circular holes. II. Indian J. Theoret. Phys. 2 (1954), 37–46.

[For part I see MR 13, 510.] In this paper the influence of stresses around a circular hole in a semi-infinite plate subjected to a couple M applied on its straight edge is discussed. *Author's summary.*

Sen Gupta, A. M. Stresses in certain thin elastic plates rotating about normal axes. Z. Angew. Math. Mech. 37 (1957), 27–34. (English, French and Russian summaries)

In this paper a direct method for solving problems of certain thin elastic plates having curvilinear boundaries and rotating steadily about normal axes is developed. To illustrate the method the problem of a rotating thin plate having a regular curvilinear polygonal boundary, in general, is solved. Discussions of the corresponding problems of thin plates with some simpler types of boundaries are given in detail. *Author's summary.*

Godfrey, D. E. R. Generalized plane stress in an elastic wedge under isolated loads. Quart. J. Mech. Appl. Math. 8 (1955), 226–236.

An account is given of the application of Tranter's method of Mellin transforms [same J. 1 (1948), 125–130; MR 10, 43] to the solution of the equations of generalized plane stress in polar coordinates. Throughout the author expresses the mean stresses and displacements in terms of the complex potentials introduced by Stevenson [Phil. Mag. (7) 34 (1943), 766–793; MR 8, 116]. This considerably shortens the formal analysis and leads to solutions for nuclei of force or couple acting at any point of an infinite wedge thereby avoiding the usual tentative approach offered by an Airy stress-function method. The solutions are derived for a wedge with stress-free or rigid boundaries. *I. N. Sneddon* (Glasgow).

Godfrey, D. E. R. Normal loading on a wedge-shaped plate. Aero. Quart. 6 (1955), 196–204.

It is shown that the equations of thin plate theory respond to the treatment introduced by the author in the paper reviewed above. A complete formal solution for the stress resultants and couples is derived and the special solutions for (a) uniform loading over a sectorial area, (b) an isolated normal force at a point on the axis of symmetry, are then deduced. *I. N. Sneddon.*

Horvay, G.; and Born, J. S. Some mixed boundary-value problems of the semi-infinite strip. J. Appl. Mech. 24 (1957), 261–268.

Rigorous and approximate (variational) solutions are given for the semi-infinite elastic strip when the short edge is subjected (a) to a quadratic shear displacement, zero normal stress, (b) to a cubic normal displacement, zero shear stress; the long edges are stress-free in both cases. The exact solutions are expansions in terms of the Fadle-Papkovich eigenfunctions obtained by use of Fourier integrals. The stress distributions are plotted and various important details, e.g. the behaviour of the solutions near the corners, are discussed.

The approximate solutions use particularly appropriate orthogonal polynomials. *R. C. T. Smith* (Armidale).

Brown, E. H. The diffusion of load from a stiffener into an infinite elastic sheet. Proc. Roy. Soc. London. Ser. A. 239 (1957), 296–310.

The use of complex variable theory to express problems in generalised plane stress is well known, but methods of finding particular solutions are available for only a limited range of problems. This paper and its sequel will develop a new technique, reducing certain problems with mixed boundary conditions to second order functional differential equations, whose solutions can be found in series form.

Exact solutions are given to three fundamental problems of the diffusion of load into an infinite two dimensional elastic sheet to which a semi-infinite elastic stiffener is continuously attached throughout its length. The first problem has a load applied to the end of the stiffener, with its line of action along the stiffener and its reactions at infinity. In the other two problems the stiffener end is unloaded but a uniform tension is applied to the sheet at infinity, in one case parallel to the stiffener, in the other perpendicular to it. Expressions for the load in the stiffener and for the direct and shear stresses in the sheet are found and plotted in non-dimensional form. (Author's summary.) *R. M. Morris* (Cardiff).

Golecki, Józef. The sphere weakened by a concentric inclusion of different elastic properties under concentrated loads. Arch. Mech. Stos. **9** (1957), 301–317. (Polish and Russian summaries)

This paper aims at the displacements and stresses in an elastic sphere with a concentric spherical inclusion of different elastic properties, which is subjected to two equal and diametrically opposite concentrated loads applied to the boundary. A solution is obtained in infinite series form corresponding to the limiting cases of (1) a spherical shell and (2) a rigid inclusion. The problem considered could have been reduced to one governed by finite and continuous surface tractions with the aid of an available solution to the corresponding problem of the solid sphere. In the present treatment, however, the singularities at the load points are not removed to any extent. *E. Sternberg* (Providence, R.I.).

Neidhardt, G. L.; and Sternberg, Eli. On the transmission of a concentrated load into the interior of an elastic body. J. Appl. Mech. **23** (1956), 541–554.

A series solution is given for stresses and displacements in an elastic body bounded by one sheet of a two-sheeted hyperboloid of revolution, subjected to an axial concentrated load at the vertex. Solution is based on the Boussinesq stress functions referred to spheroidal coordinates and corresponding known solutions appropriate to the half-space and to the circular cone are obtained as limiting cases. Numerical results are given for the normal stress on planes perpendicular to the axis of symmetry, at points on this axis. Results indicate that the influence of curvature of the boundary at the load point upon the transmission of the load into the interior of the body may be considerable. *A. E. Green* (Newcastle-upon-Tyne).

Kogan, S. Ya. On the solution of the space problem of the theory of elasticity by the alternating method of Schwarz. Izv. Akad. Nauk SSSR. Ser. Geofiz. **1956**, 307–319. (Russian)

The author solves the first fundamental problem in the theory of elasticity by use of the alternating method of Schwarz. This problem consists of finding a solution of the basic equations of elasticity in a domain D bounded by a surface on which the surface stresses are specified.

The alternating method of Schwarz may be summarized briefly as follows: if there is given a region D which is the sum or intersection of two or more regions, say D_1+D_2, then the boundary value problem is first solved in one of the component regions, say D_1. The solution so obtained determines a boundary function on that part of the boundary of D_2 which is interior to D. This together with the given exterior surface stresses gives boundary values for D_2. Then the boundary value problem is solved in D_2. This solution gives boundary values for D_1 on the interior boundary which together with the given exterior surface stresses give boundary values for D_1. The boundary value problem is again solved in D_1, etc.

The solution of the problem of elasticity is then obtained by first constructing a sequence of tensors $\{T_i\}$ by the alternating method of Schwarz and showing that $\{T_i\}$ is compact. The limit tensor T is shown to satisfy the equations of equilibrium and the Beltrami equations. Finally it is shown that T also satisfies the boundary conditions and is therefore the solution of the first boundary value problem of elasticity. *C. G. Maple.*

Eshelby, J. D. The determination of the elastic field of an ellipsoidal inclusion, and related problems. Proc. Roy. Soc. London. Ser. A. **241** (1957), 376–396.

This paper deals with two related problems in classical elasticity theory: (1) the determination of the stresses and displacements produced in a medium occupying the entire space if an ellipsoidal subregion undergoes a spontaneous deformation which would be homogeneous in the absence of the surrounding material; (2) the analogous problem for the case in which inclusion and matrix have different elastic constants, while the stresses are induced by a uniform stress-field at infinity.

In approaching his objectives, the author proceeds through a sequence of "imaginary cutting, straining, and welding operations" which are claimed to be equivalent to a formulation of (1) and (2) as standard boundary-value problems of the theory. In connection with (1), integral representations for the solution are first deduced for an inclusion of arbitrary shape and later specialized to the ellipsoid; the treatment of (2) is confined to the ellipsoidal inclusion. The method employed rests on classical potential theory and cartesian coordinates are used throughout.

In contrast, Sadowsky and Sternberg [J. Appl. Mech. **16** (1949), 149–157; MR **10**, 760] used ellipsoidal coordinates, to reach closed solutions in terms of elliptic functions and integrals for a special case of (2) in which the infinite medium possesses an ellipsoidal cavity. By similar means, Robinson [J. Appl. Phys. **22** (1951), 1045–1054; MR **14**, 111] whose work is referred to by the author, disposed of the case in which the inclusion has elastic properties different from those of the surrounding material and the entire body is subjected to a uniform change in temperature. Since the present results are largely formal and do not yield an explicit characterization of the complete stress and displacements fields, the author's claim of comparative simplicity and directness of his approach may be open to misinterpretation.
E. Sternberg (Providence, R.I.).

Chandra Das, Sisir. On the elastic distortion of a cylindrical hole by localized axial shears on the inner boundary. Indian J. Theoret. Phys. **1** (1953), 41–46.

The effect of a localized axial shear as produced by a piston head on the inner surface of an infinitely long cylindrical hole in an elastic solid has been considered in this paper. *Author's summary.*

Chakravorty, J. G. On the stress distribution in an infinite elastic solid of a transversely isotropic material with a cylindrical hole due to a localised axial shear. Indian J. Theoret. Phys. **3** (1955), 119–124.

In the present paper the author considers the problem

of localised axial shear on the boundary of a cylindrical hole in an infinite solid of a transversely isotropic material. The axis of the cylindrical hole is here parallel to the axis of symmetry of the material. The corresponding problem for isotropic material has been solved previously by Chandra Das [see the paper reviewed above]. For the solution of the present problem use has been made of the two functions introduced by Elliott [Proc. Cambridge Philos. Soc. **44** (1948), 522–533; MR **10**, 167], in terms of which he has expressed the stresses and displacements, but instead of using the expressions of Elliott the author finds it more convenient to use expressions in terms of the elastic constants. The problems of stress distribution in an infinite circular cylinder of a transversely isotropic material and of an infinite solid with a cylindrical hole under localised hydrostatic pressure on a boundary and under localised shear on a boundary perpendicular to the axis of the boundary have also been considered by the present author. The paper concerned with this problem is reviewed below. {There are some small misprints in the paper, e.g. on p. 124.} *R. Gran Olsson* (Trondheim).

Chakravorty, J. G. Concentration of stress in the neighborhood of a small spherical hole on the axis of a cylinder under torsion of transversely isotropic material. J. Assoc. Appl. Phys. **4** (1957), 13–20.

In this paper the author treats the problem indicated in the title by reducing it to an equivalent problem for an isotropic medium. He is led in this way to the isotropic torsion problem for a shaft with spheroidal cavity. This is one of a number of cavity problems which have been treated in the literature [see, e.g., A. Weinstein, Quart. Appl. Math. **10** (1952), 77–81; MR **14**, 221; Payne, J. Soc. Indust. Appl. Math. **1** (1953), 53–71; Proc. Cambridge Philos. Soc. **50** (1954), 466–473; MR **15**, 267, 1004].

L. E. Payne (College Park, Md.).

Chandra Das, Sisir. On the effect of shearing stresses applied on the boundary of a circular hole in a semi-infinite plate. Indian J. Theoret. Phys. **1** (1953), 79–86.

In this paper bipolar co-ordinates have been used to determine stresses due to applied tangential forces on the inner boundary of a circular hole near the straight edge of a semi-infinite plate. *Author's summary.*

Chandra Das, Sisir. On the stresses due to a small spherical inclusion in an elastic solid under uniform shearing stress. Indian J. Theoret. Phys. **1** (1954), 171–182.

In this paper, the concentration of stresses in an isotropic elastic solid due to a small spherical inclusion under uniform shearing stress has been found. The results are discussed for several particular cases.

Author's summary.

Chandra Das, Sisir. On the stresses in twisted composite spheres and spheroids. Canad. J. Phys. **35** (1957), 811–817.

The author obtains closed solutions for the stresses arising in the pure torsion of a composite elastic body which is made up of a spherical (or spheroidal) core ideally bonded to a spherical (or spheroidal) shell with different elastic properties. The loading, in either case, consists of one particular distribution of shearing traction which, on each of two hemi-spheres (or hemi-spheroids) of the outer boundary, is statically equivalent to a couple.

E. Sternberg (Providence, R.I.).

Galimov, K. Z. Conditions of continuity of deformation of a surface for arbitrary bending and strain. Kazan. Gos. Univ. Uč. Zap. **113**, no. 10 (1953), 161–164. (Russian)

Išlinskiĭ, A. Yu. On the residual stresses under torsion. Ukrain. Mat. Ž. **4** (1952), 155–167. (Russian)

Klein, Bertram. A simple method of matric structural analysis. J. Aero. Sci. **24** (1957), 39–46.

An ingenious general method of matric structural analysis is presented in which the set up time for solution is reduced while the matrix required for solution is increased. Author points out that use of automatic computing equipment for solution of matrix can make this an overall optimum method. In an example the method is seen to be powerful and accurate. It yields deflections (and hence influence coefficients) as well as forces.

S. Levy (Washington, D.C.).

Naghdi, P. M. Note on the equations of shallow elastic shells. Quart. Appl. Math. **14** (1956), 331–333.

Generalisation of the differential equations of the linear theory of thin shallow elastic shells so as to include the effect of transverse shear deformation in a manner which corresponds to what has previously been done for flat plates. *E. Reissner.*

Sapondžyan, O. M. Bending of a clamped plate under a load uniformly distributed over the surface of a circle. Akad. Nauk Armyan. SSR. Izv. Fiz. Mat. Estest. Tehn. Nauki **9** (1956), no. 5, 61–75. (Russian. Armenian summary)

Evan-Iwanowski, R. M. Stress solutions for an infinite plate with triangular inlay. J. Appl. Mech. **23** (1956), 336–338.

The method of N. I. Muskhelisvilli for solving two-dimensional problems in elasticity by means of the functions of complex variable has been extensively applied to the problems of infinite plates with inlays or openings. This method was essentially developed by E. Goursat [Bull. Soc. Math. France **26** (1898), 236–237]; G. V. Kolosov [Z. Math. Phys. **62** (1914), 384–409]; Muskhelisvilli [Z. Angew. Math. Mech. **13** (1933), 264–282] and has in the present paper been applied to the following examples: a) An infinite, homogeneous and isotropic plate with a rigid inlay of triangular shape under uniform tension at infinity; b) a concentrated force; and c) a moment acting on a triangular inlay in an infinite plate. All these problems are second boundary-value problems, i.e. the displacements are prescribed on the boundary. The mapping function used in this paper is $z=w(\zeta)=K(\zeta+n/\zeta^2)$, where K is real, $0\leq n\leq\frac{1}{3}$ and real, and it maps an exterior of a triangle with rounded corners in the z-plane into an exterior of a unit circle in the ζ-plane. — Several examples relate to the general use of mapping functions of the type $z=w(\zeta)=K(\zeta+n/\zeta^m)$, $K>0$, $0\leq n\leq 1/m$, where m is a positive integer given in the papers of G. N. Savin [Trudy Dnepropetrovsk. Inž.-Stroit. Inst. **10** (1936)], A. N. Dinnik, A. B. Morgaevski and G. N. Savin [Conference on the character of underground pressure, Izdat. Akad. Nauk SSSR, 1938, pp. 7–55], and G. S. Shapiro [Trudy Leningrad. Politehn. Inst. no. 3 (1941), 184–199]. In the author's opinion there is no need to publish another description of the method itself or to give a derivation of intermediate functions and expressions entering into these problems.

R. Gran Olsson (Trondheim).

Parkus, H. Membranspannungen in der schiefen Kreiskegelschale. Österreich. Ing.-Arch. **9** (1955), 196–199.

The equations of the general linear theory of thin shells [in the form presented by the author in Österreich. Ing.-Arch. **4** (1950), 160–174; **6** (1951), 30–35; MR **11**, 701; **13**, 601] are specialized to the case of the membrane stress state in an oblique circular conical shell. The equilibrium equations for the determination of the membrane stress resultants are presented. *S. R. Bodner.*

Muster, D. F.; and Sadowsky, M. A. Bending of a uniformly loaded semicircular plate simply supported around the curved edge and free along the diameter. J. Appl. Mech. **23** (1956), 329–335.

In the present paper, within the limits of classical plate theory, the deflections, bending moments and shear forces of a semicircular elastic plate under the action of a uniform lateral load are obtained. The plate is simply supported around the curved edge and free along the diameter. The solution obtained in series form, although singular at the corners, is otherwise analytic and amenable to numerical evaluation by electronic calculators. The numerical results are compared to those obtained experimentally by E. L. Robinson [Gen. Elec. Co. Rep. no. **16380** (1923)] and A. Huggenberger [Forschungsarb. Gebiete Ingenieurswesens no. **280** (1926)]; A. M. Wahl [Trans. A.S.M.E. **54** (1932), APM 311–320]. The agreement between theoretical and experimental results seems to be reasonable, when account is taken to the manner in which the empirical results were obtained. The principal conclusions may be summarized as follows: 1) The numerical results agree reasonably well with the available evidence. 2) The solution has singularities at the corners corresponding to concentrated forces. 3) The concentrated reactions at the corners each have magnitude equal to about 22 per cent of the total load on the plate. 4) The maximal deflection of the semicircular plate is 2.3 times the maximum of deflection of a uniformly loaded, simply supported circular plate. 5) The maximum bending stress of the semicircular plate (at $r=0$) is 1.5 times the maximum bending stress of a uniformly loaded, simply supported circular plate. *R. Gran Olsson* (Trondheim).

Rüdiger, D. Die strenge Theorie der Faltwerke konstanter Krümmung. Österreich. Ing.-Arch. **11** (1957), 5–20.

The theory is concerned with the analysis of systems of concentric circular ring plates which are stressed in transverse bending as well as in their own planes and which are connected through circular connecting rings. The solution of the problem of the individual plate as well as the problem of the individual ring can be considered as known. To be established is a systematic procedure for the analysis of any given configuration consisting of a number of the individual structural components. As an example of application of his general formulas the author considers a single circular ring plate supported by two unequal edge rings and acted upon by a rotationally symmetric local distribution. *E. Reissner.*

Eringen, A. C. Response of beams and plates to random loads. J. Appl. Mech. **24** (1957), 46–52.

The paper is concerned with the probabilistic problem of the vibrating elastic beams and plates under stochastic loading. Generalized harmonic analysis developed by Wiener [Acta Math. **55** (1930), 117–258] is applied to obtain the cross-correlation functions for displacements, stresses, etc. in terms of a particular type of cross-correlation function of pressure loading. The author then determines probable number of times per unit time the random displacements or stresses will exceed a given value using a method due to Rice [Bell System Tech. J. **23** (1944), 282–332; **24** (1945), 46–156; MR **6**, 89, 233]. Detailed results are obtained for the case of simply supported bars, cantilever bars, clamped circular plates and simply supported rectangular plates. *E. T. Onat.*

Knowles, James K.; and Reissner, Eric. A derivation of the equations of shell theory for general orthogonal coordinates. J. Math. Phys. **35** (1957), 351–358.

In a previous paper [Reissner, Amer. J. Math. **63** (1941), 177–184; MR **2**, 272] the equations of the title were derived assuming that the coordinate curves on the middle surface were curvature lines and that these lines, together with the normals to the surface, form an orthogonal system of space coordinates. All other orthogonal systems on the middle surface, together with the normals to the surface, form non-orthogonal coordinate systems. Present derivation generalizes former and avoids use of oblique space coordinates and the notion of stress and strain referred to these coordinates. *H. D. Conway.*

Hellman, Olavi. Beiträge zur allgemeinen Schalentheorie mit besonderer Berücksichtigung der Randstörungen und der flachen Schalen. Ann. Acad. Sci. Fenn. Ser. A. I. no. **213** (1955), 71 pp.

After some general remarks on thin shell theory, of which some appear questionable (examples: (1) the results of a theory of thin shells are applicable only provided not only the thickness but also the displacements are small; (2) the most important development of general shell theory has its beginning in 1944) the author considers a tensor-analytical formulation of linear shell theory and of certain boundary layer approximations associated with it. Throughout, it is not clear where the treatment is an exposition of known results and where it is thought that additions to previous knowledge are involved. *E. Reissner* (Cambridge, Mass.).

Alekseev, S. A. Postcritical work of flexible elastic plates. Prikl. Mat. Meh. **20** (1956), 673–679. (Russian)

The problem of the behaviour of elastic plates loaded in their planes by forces applied at the boundaries using von Karman's equations for large deflections of plates is discussed in this paper. A general method of successive approximations to the solution of the problem is developed and an example of a rectangular simply supported plate loaded by uniformly distributed end loads acting on two opposite sides is solved in detail. *E. P. Popov.*

★ **Goodman, L. E. Circular-crested vibrations of an elastic solid bounded by two parallel planes.** Proceedings of the First U.S. National Congress of Applied Mechanics, Chicago, 1951, pp. 65–73. The American Society of Mechanical Engineers, New York, N. Y., 1952.

Axially symmetric vibrations are considered in cylindrical coordinates. Semi-inverse method of attack is used by assuming radial functions to be Bessel functions. Radial displacements given by even functions of the axial coordinate determine extensional vibrations, and odd functions, flexural vibrations. Tables of wave length and frequency are given for various values of the para-

meters. Displacement fields are studied, and it is shown that certain solutions provide exact modes of vibration for certain finite axially symmetric bodies. The limit at large radius is shown to give the plane strain modes which have been considered previously. *E. H. Lee.*

Wallisch, W. Einfluss der Schubverzerrung auf die Eigenschwingungen von Platten. Z. Angew. Math. Mech. 36 (1956), 291–293.

Brief outline of an asymptotic solution procedure for the equations of three-dimensional elasticity applied to the problem of transverse plate vibrations. Evaluation of the merits of the procedure will require the study of a more detailed report than that given in the present paper. *E. Reissner* (Cambridge, Mass.).

Chakravorty, J. G. Vibrations of a circular cylinder of transversely isotropic material. Proc. Nat. Inst. Sci. India. Part A. 22 (1956), 220–227.

Kil'čevskiĭ, N. A. Approximate method of solution of certain dynamical problems of the theory of elasticity. Akad. Nauk Ukrain. RSR. Prikl. Meh. 1 (1955), 251–267. (Ukrainian. Russian summary)

A method of approximate solution of elastodynamic equations which permit the solution of certain cases of forced vibrations of elastic bodies is presented. The basis of the method is a transformation of the equilibrium equations of the theory of elasticity. The author uses a simple transformation of the statical dimensionless Descartes' coordinates into dynamical ones. This transformation gives the relations between the static and dynamic problems of the theory of elasticity. The error of the approximate solution is discussed, as well as the conditions which the stresses on the plane boundary of the semispace satisfy. *D. Rašković* (Belgrade).

Warburton, G. B. The vibration of rectangular plates. Proc. Inst. Mech. Engrs. 168 (1954), 371–381; discussion 381–384.

The main body of the paper deals with the free transverse vibrations of rectangular elastic, isotropic, uniformly thick plates, making use of the classical small deflection theory. Some fifteen sets of boundary conditions are handled, including combinations of free, freely supported, and clamped edges. The analysis employs the Rayleigh-Ritz method; the assumed waveform is developed from those of beams with related end conditions, the coefficients being chosen to minimize the Rayleigh quotient. Some, though not all, of these problems have been investigated previously by various authors; however, the present paper includes a very extensive table of computed values, and numerous references are given to earlier work. The author estimates that the error between the true frequencies and his approximate results is less than one percent in all but a few cases. Considerable attention is also given to the dependence of the nodal line pattern on the dimensions of the plate for the two cases in which all four edges are either clamped or free. The pattern is quite sensitive to changes in the length-width ratio when this ratio is near one, and this relationship is treated in detail.

The discussion evoked by the paper centered about the possibility of applying this technique to vibrations of plates which are either non-rectangular or anisotropic. *W. E. Boyce* (Troy, N.Y.).

Deresiewicz, H.; and Mindlin, R. D. Axially symmetric flexural vibrations of a circular disk. J. Appl. Mech. 22 (1955), 86–88.

The frequency equation for axially symmetric flexural vibrations of a thin, isotropic, elastic disk with a traction-free boundary is derived, including the effects of rotatory inertia and shear deformation. The frequency spectra in the neighbourhood of the frequency, $\bar{p}$, of the fundamental thickness-shear mode of an infinite plate are investigated and compared with the corresponding spectra predicted by the classical theory of plates. The most striking difference between the two theories occurs at frequencies above $\bar{p}$. *G. B. Warburton* (Edinburgh).

Piszczek, Kazimierz. The influence of the curvature of an originally curved bar on the resonance regions of plane form of bending. Arch. Mech. Stos. 9 (1957), 155–189. (Polish and Russian summaries)

The author investigates the bending and torsional vibrations of a curved bar subjected to a driving moment

$$M(t)=M_0+M_1 \sin \omega t.$$

The bar is assumed to have a constant radius of curvature R, and its two ends are either hinged or built-in. The author shows that in both cases the equation of motion can be expressed as

$$\frac{d^2y}{d\tau^2}+\mu P(\tau)y=0,$$

where y is a two-element displacement vector, P is an appropriate two-by-two matrix, μ specifies the frequency, and τ is a time-like variable. The stability of the solutions of this equation is analyzed, and resonance regions in the $M_1-\omega^{-2}$ plane are described. Numerical results are given in particular cases. Among other things, the author shows that there is a value of R for which the free vibrations of the system are uncoupled; he further concludes, from his numerical computations and without a general proof, that this value of R produces the minimum resonance region for a given M_1. Incidentally, the reader will note that Figs. 7 and 11 have been interchanged. *W. E. Boyce.*

Vol'vič, S. I. Problem of stability in the theory of elasticity. Trudy Saratov. Avtomobil'nodorož. Inst. 13 (1955), 3–15. (Russian)

Nash, W. A. Buckling of initially imperfect cylindrical shells subject to torsion. J. Appl. Mech. 24 (1957), 125–130.

Existing treatment [T.-T. Loo, Proc. 2nd U.S. Nat. Congress Appl. Mech., Ann Arbor, 1954, Amer. Soc. Mech. Engrs., New York, 1955, pp. 345–357] of torsional bucklings of circular cylindrical shells using large-deflection theory has been extended by considering the effects of having either clamped or simply supported edges, and by minimizing the potential energy with respect to all free parameters instead of taking the value of some of them from small-deflection theory. The effects of this greater generality are illustrated by means of numerical calculations of parameter values and of load-deformation characteristics. Of importance is that the present calculations show a greater drop of resistance during the past-buckling stage than was found in the earlier work referred to above. *E. Reissner* (Cambridge, Mass.).

Lee, Lawrence H. N.; and Ades, Clifford S. Plastic torsional buckling strength of cylinders including the effects of imperfections. J. Aero. Sci. **24** (1957), 241–248, 264.

The torsional buckling stress of a thin-walled cylinder in the plastic range is investigated. On account of experience with other buckling investigations for plates and shells in the plastic range, stress-strain relations of deformation type are used in combination with Shanley's concept of simultaneously increasing load and buckling deflection [J. Aero. Sci. **14** (1947), 261–268]. Energy solution for cylinders of finite lengths and more exact solution for infinitely long cylinders are compared; the agreement is reasonable. The effect of large deflections (post-buckling behaviour) and of initial imperfections are analyzed by means of Donnell's method in which proportionality between initial deviation from perfect shape and buckling shape is assumed [Trans. A.S.M.E. **56** (1934), 795–806]. For two groups of experimental results used for comparison, the effects of geometrical imperfections are negligible in the plastic range according to theory. This result is borne out by the experiments which show good agreement with theoretical buckling stress; the experimental values range from 82 to 102 percent of theoretical values for aluminium alloy cylinders. *W. T. Koiter* (Delft).

Sokołowski, Marek. The stability of an orthotropic infinitely long plate with ribs. Arch. Mech. Stos. **8** (1956), 507–538.

Paper gives an extensive investigation of the problem of combined bending and compression of the title problem, the ribs being prismatic bars rigidly connected to the plate. The structure is divided into elements consisting of beams and plates, beam and plate theory being used for the solution. *H. D. Conway* (Ithaca, N.Y.).

Nowacki, Witold; and Sokołowski, Marek. Certain stability problems of rectangular plates. Arch. Mech. Stos. **9** (1957), 109–124. (Polish and Russian summaries)

Rectangular isotropic plates simply supported on two opposite edges and having different boundary conditions along the two remaining edges are considered, the stresses being in general functions of x, y. It is simplest to explain their method by taking a particular case.

A plate simply supported at $x=0$, a, and clamped at $y=0$, b is regarded as portion of an infinite strip simply supported at $x=0$, a, subject to periodic stresses (with period b) and to additional forces $r(x)$ and $t(x)$ perpendicular to the plate and acting along the segments $y=\pm ib/2$ ($i=0, 1, 2, \cdots$). The value of these additional forces is chosen so as to make the deflection w of the plate vanish along their line of action.

The differential equation of this problem is converted to an integro-differential equation by means of Green's function for the biharmonic equation. Substituting a double Fourier series for w yields finally an infinite system of linear equations which can be treated by the familiar method of segments.

As examples several problems already solved by other methods are treated.

The method is obviously of great generality.

R. C. T. Smith (Armidale).

Ryabova, E. V. The problem of a wedge striking a thread. Vestnik Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him. **11** (1956), no. 1, 57–62. (Russian)

The author presents a general discussion of the vibration of an elastic thread (wire) under the influence of the striking action of a rigid wedge. A system of three partial differential equations contains the explicit solution to the problem in question. By virtue of the fact that these equations are inhomogeneous, the author proposes to solve them by means of successive approximations. Some boundary conditions are also deduced from geometrical considerations. The method of approximation adopted is shown to converge in general; as a matter of fact, she claims that only the first-order approximation yields high accuracy. It seems to the present reviewer, however, that the author's analysis is rather scanty and important details are lacking. And, therefore, the problem set forth is virtually still unsolved. *K. Bhagwandin.*

Arutyunyan, N. H.; and Čobanyan, K. S. On torsion of prismatic rods composed of different materials, taking account of creep. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1956**, no. 6, 16–34. (Russian)

Prokopovič, I. E. On the solution of the plane contact problem taking account of creep. Prikl. Mat. Meh. **20** (1956), 680–687. (Russian)

Finzi, Leo. Sulla evoluzione delle frontiere nei problemi elastoplastici tridimensionali. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. **90** (1956), 528–535.

Trifan, D. On a complementary energy principle. J. Math. Mech. **6** (1957), 87–90.

In a previous paper [Quart. Appl. Math. **13** (1955), 337–339; MR **17**, 321] a minimum principle for strain rates was given for a material obeying the stress-strain relations proposed by the author. The present paper is concerned with the development of a minimum principle for the rates of stress deviation. *E. T. Onat.*

Hopkins, H. G. On the plastic theory of plates. Proc. Roy. Soc. London. Ser. A. **241** (1957), 153–179.

The self-contained paper offers an excellent outline of the theory of small bending deformations of plates that consist of a rigid, perfectly plastic material obeying Tresca's yield condition and the associated flow rule. Field equations are established that involve the bending and twisting moments and rates of curvature and twist. These are transformed to a curvilinear coordinate system whose coordinate curves indicate the principal directions of curvature. Discontinuities in the field variables are discussed in considerable detail. While the author presents the general theory, he does not solve any specific boundary value problems. *W. Prager* (Providence, R.I.).

Naghdi, P. M. Stresses and displacements in an elastic-plastic wedge. J. Appl. Mech. **24** (1957), 98–104.

The problem of a thick wedge, subject to uniform normal pressure of gradually increasing intensity along one face, is investigated under conditions of plane strain. The analysis applies to elastic, perfectly-plastic, incompressible material obeying the Prandtl-Reuss equations. Let β denote the wedge angle. The complete solution to this problem is obtained for $0<\beta<\frac{1}{2}\pi$, the known elastic and plastic-rigid solutions simply occurring as limiting cases of the general solution. Detailed numerical results are given for the case $\beta=\frac{1}{2}\pi$ and for various positions of the plastic-elastic interface. The inherent simplicity of the problem is due to the fact that the conditions of the problem involve no fundamental

linear dimension (the wedge being supposed indefinite in extent). Plastic deformation is initiated simultaneously along both faces of the wedge. Subsequently, there are two straight plastic-elastic boundaries passing through the wedge apex and being symmetrically disposed with respect to the faces, these boundaries gradually moving in towards the wedge centre line with increasing plastic deformation. All stresses are simply dependent upon an angular co-ordinate together with a single monotonic parameter (here taken as an angular co-ordinate specifying the position of either one of the plastic-elastic boundaries) associated with the progress of deformation.

H. G. Hopkins (Sevenoaks).

Gerard, George. Plastic stability theory of thin shells. J. Aero. Sci. **24** (1957), 269–274.

Plastic buckling of thin shells of constant, unequal principal radii is analyzed on the basis of approximate equilibrium equations of the Donnell type, and the simple deformation theory of plasticity. The assumption of no plastic unloading, corresponding to buckling with increasing load (Shanley-type buckling) is used. Plasticity reduction factors, to be applied to results for elastic buckling, are calculated for spheres under external pressure, circular cylinders in torsion, and circular cylinders under lateral pressure. *B. Budiansky.*

Sokolovskiĭ, V. V. On equations of plastic flow in the boundary layer. Prikl. Mat. Meh. **20** (1956), 328–334. (Russian)

Sternberg, E.; and McDowell, E. L. On the steady-state thermoelastic problem for the half-space. Quart. Appl. Math. **14** (1957), 381–398.

Problem is treated within the three-dimensional classical theory of elasticity and is approached by the method of Green. Authors show that the stress field induced by an arbitrary distribution of surface temperatures on the plane boundary of the half-space, is plane and parallel to the boundary. If the surface temperature is prescribed arbitrarily over a bounded region B of the boundary and is otherwise constant, authors reduce problem to the determination of Boussinesq's three-dimensional logarithmic potential for a disk in the shape of the region B, whose mass density is equal to the given temperature. A solution in terms of complete and incomplete elliptic integrals is given for a circular region B at uniform temperature. Solutions in terms of elementary functions are given for a hemispherical distribution of temperature over a circular region, as well as for a rectangle at constant temperature. *A. E. Green.*

Biot, M. A. Influence of thermal stresses on the aeroelastic stability of supersonic wings. J. Aero. Sci. **24** (1957), 418–420, 429.

In an earlier paper [same J. **23** (1956), 237–251, 271], the author dealt with the problem of the divergence of supersonic wings including chordwise bending. Several different exploratory analytical approaches to the problem were outlined, the most powerful being an approximate solution for a thin wing with a double-wedge profile, which accounts for the effects of nonlinear plate deflections and Poisson's ratio (anticlastic chordwise curvature).

The present paper employs a similar approach for the study of the influence of thermal stresses on the aeroelastic stability of supersonic wings. Significant spanwise membrane stresses are now induced by virtue of both anticlastic and thermal effects. Thermal effects are assumed to arise from a parabolic temperature distribution over the chord; spanwise variations are ignored. The membrane stresses and aerodynamic loading in turn result in a chordwise curvature of the wing, which is calculated on an approximate, average basis; this chordwise curvature is then responsible for a portion of the aerodynamic loading.

The wing is next treated as a bent and twisted beam, with the twist and lateral deflection at the midchord used as displacement coordinates. This leads to a differential equation of the fourth order for the wing twist along the span, which is used to study the stability of a root-cantilevered wing.

For incipent instability, i.e., for very small wing deformations, the results show that during a period of wing heating the possible range of unstable configurations is reduced; the reverse is true during a cooling period. With regard to the torsional stiffness required to prevent instability, thermal effects can be either stabilizing or destabilizing depending on the design parameters.

In the reviewer's opinion, the pair of papers discussed here represent an illuminating contribution to the aeroelastic literature. Their results should find valuable applications in interpreting and extrapolating test results, as well as in design appraisals of simple wing configurations.

M. Goland (San Antonio, Tex.).

Hieke, Max. Eine indirekte Bestimmung der Airyschen Fläche bei unstetigen Wärmespannungen. Z. Angew. Math. Mech. **35** (1955), 285–294. (English, French and Russian summaries)

Hieke, Max. Die Deutung bestimmter ebener Eigenspannungszustände als Membranspannungen. Z. Angew. Math. Mech. **37** (1957), 177–191. (English, French and Russian summaries)

In linear thermoelasticity, one can define thermal strain as the Cartesian tensor $\varepsilon_{ij}=e_{ij}-\alpha t\delta_{ij}$, where e_{ij} is the actual strain, t is temperature measured on a suitable scale and α is a material constant. In general, ε_{ij} is not expressible as the symmetric part of a displacement gradient, hence is not interpretable as an infinitesimal strain in Euclidean space. In special cases, the author attempts to interpret ε_{ij} in terms of two dimensional surfaces not applicable to a plane. *J. L. Ericksen.*

★ **Chu, Boa-Teh. Thermodynamics of elastic and of some visco-elastic solids and non-linear thermoelasticity.** Tech. Rep. No. 1, Nonr-562(20), Division of Engineering, Brown University, Providence, R. I., July, 1957. ii+44 pp.

This report contains a clear and detailed account of the general principles governing finite thermoelastic deformations. The treatment is not confined to adiabatic or isothermal processes; arbitrary changes of the thermodynamic variables may occur. Apart from the equation of conservation of mass, the equations of motion, and the energy equation, the theory is based on the following three principles:

(a) There is an "equation of state"

$$A=A(\mathbf{C}, T), \tag{1}$$

stating that the free energy A is a function of the strain tensor $\mathbf{C}$ and the temperature T only.

(b) There is a law of heat conduction

$$\mathbf{q}=\mathbf{q}(\mathbf{C}, T, \operatorname{grad} T) \tag{2}$$

stating that the heat flux **q** is a function of the variables indicated. A discussion of this general law will be given in a forthcoming paper by R. S. Rivlin and the author.

(c) For an infinitesimal deformation of a strained body the "first law of thermodynamics"

$$S\delta T=\delta A-\delta W \tag{3}$$

holds, where S is the entropy and δW the work done. The details of the theory are mostly confined to the case of isotropic bodies. In the last chapter, the author generalizes his theory to a special class of visco-elastic materials. He assumes that the stress tensor is the sum of a reversible and an irreversible part, the equation (3) being valid only for the work done by the reversible stress and the irreversible stress being a linear function of the rate of deformation. [Reviewer's remark: The term "first law of thermodynamics" for (3) is misleading. The assumption (3) is not satisfied for non-elastic materials, and it has no relation to the conservation of energy. The author has not yet overcome the difficulties discussed by Truesdell [pp. 168–171 of J. Rational Mech. Anal. **1** (1952), 125–300; MR **13**, 794.] *W. Noll* (Pittsburgh, Pa.).

Ting, Tsuan Wu; and Li, James C. M. Thermodynamics for elastic solids. General formulation. Phys. Rev. (2) **106** (1957), 1165–1167.

Gibbs, Duhem, Schiller, and others in the last century worked out the transformation formulae for thermodynamic potentials when the caloric equation of state involves an arbitrary finite number of parameters. The authors give results of this kind in the context of linear elasticity theory. *C. Truesdell* (Bloomington, Ind.).

Mirzadžanzade, A. H.; and Abbasov, A. A. Approximate solution of the problem of heat exchange in the case of motion of visco-plastic fluid in a cylindrical tube. Akad. Nauk Azerbaĭdžan. SSR. Dokl. **12** (1956), 155–161. (Russian. Azerbaijani summary)

See also: Partial Differential Equations: Ericksen; Campanato; Košelev; Gould. **Quantum Mechanics:** Brillouin.

Structure of Matter

Willis, B. T. M. An optical method of studying the diffraction from imperfect crystals. I. Modulated structures. Proc. Roy. Soc. London. Ser. A. **239** (1957), 184–191.

Very perfect two-dimensional gratings to simulate distorted and undistorted crystals are prepared photographically, and their optical diffraction patterns, equivalent to the square of the modulus of the Fourier transform of the grating pattern, are observed. Satisfactory agreement between theory and experiment is obtained for sinusoidal modulation of (i) lattice parameter, (ii) atomic scattering factor, and (iii) both.

A. J. C. Wilson (Cardiff).

Willis, B. T. M. An optical method of studying the diffraction from imperfect crystals. II. Crystals with dislocations. Proc. Roy. Soc. London. Ser. A. **239** (1957), 192–201.

An extension of the process described in the previous abstract to edge and screw dislocations.

A. J. C. Wilson (Cardiff).

Norman, N. The Fourier transform method for normalizing intensities. Acta Cryst. **10** (1957), 370–373.

The intensity of X-ray diffraction spectra is usually measured on a relative scale. A method of converting these intensities to an absolute scale is described. An electron distribution function $D(r)$ is defined such that $D(r)$ is the Fourier transform of $I(r^*)$, the intensity of the coherently scattered radiation:

$$D(r)=\int \rho(u)\rho(u+r)du=\int I(r^*)\exp[-2\pi ir^*]dr^*, \tag{1}$$

where, $\rho(u)$ and $\rho(u+r)$ are electron densities at points separated by the vector r, and r^* is a scattering vector.

Two additional equations are given:

$$KI_{\text{obs.}}=I_{\text{at.}}+I_m+I_{\text{inc.}}-I_0, \tag{2}$$

where K is the normalization constant, $I_{\text{obs.}}$ the observed relative intensity, $I_{\text{at.}}$ the structure-independent absolute intensity, I_m the structure-dependent absolute intensity, and I_0 the absolute intensity around $r^*=0$, corresponding to a uniform distribution of scattering matter (I_0 is very small);

$$I_{\text{at.}}(s)+I_{\text{inc.}}(s)=NB(s), \tag{3}$$

where $s=2\pi|r^*|$, N is the number of scattering units present, and $B(s)$ the background, and can be calculated when the chemical composition of the material is known.

From these three equations, and assuming that the observations extend over the whole of reciprocal space, and that the electron density between neighboring atoms is zero, the following general formula for K is derived:

$$\frac{K}{N}=\frac{\int B(s)dr^*-V_0^{-1}(\sum_{j=1}^{n}Z_j)^2}{\int I_{\text{obs.}}(r^*)dr^*} \tag{4}$$

where n is the number of atoms in scattering unit, V_0 the volume of the n atoms in scattering unit, and Z_j the atomic number of jth atom.

Formulae for the special cases of liquid, fiber, and crystal scattering are derived from (4).

W. Macintyre (Boulder, Colo.).

Kanzaki, H. Point defects in face-centred cubic lattice. I. Distortion around defects. Phys. and Chem. Solids **2** (1957), 24–36.

An expression for the potential energy of a defect in a real crystal is obtained. It is given in terms of the forces required to produce the same defect in an ideal perfect crystal, and the displacements produced in a lattice point in a perfect crystal by these forces. The expression is:

$$U-U_0=-\sum_{1}\sum_{\alpha}F_\alpha(r^1)\cdot\xi_\alpha(r^1)+\tfrac{1}{2}\sum_{11'}\sum_{\alpha\beta}A_{\alpha\beta}(11')\cdot\xi_\alpha(r^1)\cdot\xi_\beta(r^{1'}),$$

where r^1=position vector of an atom in the original undisplaced perfect lattice, $\xi(r^1)$=displacement of the atom, and $\xi_\alpha(r^1)$ are its rectangular components, $F_\alpha(r^1)$ are the rectangular components of the external force ($F(r^1)$) applied to the atom, U=potential energy of perfect lattice distorted under force $F(r^1)$, U_0=potential energy of undistorted lattice, and, $A_{\alpha\beta}(11')=[\partial^2U/\partial x_\alpha(r^1)\cdot\partial x_\beta(r^{1'})]_0$, where $x_{\alpha,\beta}(r^1)$ are the rectangular components of r^1, ($\alpha, \beta=1, 2, 3$).

From this equation the equilibrium values of the displacements are obtained by applying the condition $\partial U/\partial\xi_\alpha(r^1)=0$, which yields the following set of equations

$$F_\alpha(r^1)=\sum_{1}\sum_{\beta}A_{\alpha\beta}(11')\cdot\xi_\beta(r^{1'}). \tag{1}$$

The displacements $\xi(r^1)$ are then expanded in a series of

plane waves,

$$\xi(r^1)=\sum_q Q(q)\cdot\exp(iq\cdot r^1), \tag{2}$$

where q is the wave vector and $Q(q)$ are the normal coordinates of the dynamical theory of crystal lattices [M. Born and K. Huang, Dynamical theory of crystal lattices, Oxford, 1954]. Now, if

$$V_{\alpha\beta}(q)=\sum_{(1-1')} A_{\alpha\beta}(11')\cdot\exp-[iq(r^{1'}-r^1)],$$

then the equilibrium equations (1) can be written

$$G_\alpha(-q)=N\sum_\beta V_{\alpha\beta}(-q)\cdot Q_\beta(q) \tag{3}$$

where,

$$G_\alpha(-q)=\sum_1 F_\alpha(r^1)\cdot\exp-(iq\cdot r^1). \tag{4}$$

By solving equations (3) to obtain $Q(q)$, the displacements can be calculated from equations (2).

In the case of a face-centered cubic crystal the interaction potential $\phi(r)$ between two atoms a distance r apart is given by the expression $\phi(r)=B/r^{10}-C/r^6$, where B and C are constants. The values of B and C are worked out for the case of argon. The forces $F_\alpha(r^1)$ are related mathematically to the interaction potential $\phi(r)$ and the unit cell constants. The values of $F_\alpha(r^1)$ obtained in this way may be substituted in equations (4) to give the values of $G_\alpha(-q)$.

The specific cases of substitution of an inert gas atom in the argon crystal, and the introduction of a vacancy into the argon crystal are treated in detail. The magnitude of the distortion produced in the neighborhood of such substitutional defects is calculated. *W. Macintyre.*

Kanzaki, H. Point defects in face-centred cubic lattice. II. X-ray scattering effects. Phys. and Chem. Solids 2 (1957), 107–114.

Mathematical techniques for describing the distortions due to defects in face-centered cubic crystals were developed in the paper reviewed above. Using these results the following result is derived, which describes the distortions, induced by defects of several kinds, occurring throughout the crystal:

$$\xi(r^1)=\sum_s\sum_{r_{ds}^1}\sum_q Q_s(q)\cdot\exp iq(r^1-r_{ds}^1), \tag{1}$$

where s is the number of (kinds of) defects present, and the other variables have the same meaning as in part I.

In the case where the concentration of defects in the crystal is small the radiation (A_{sc}) scattered by the defects is given by:

$$A_{sc}=f_0\sum_{r^1}\exp iK\cdot[r^1+\xi(r^1)], \tag{2}$$

where K=difference of wave vectors of incident and scattered X-ray beam, and f_0=scattering factor of crystal atoms, and other variables are as in part I.

On expanding $\exp[iK\cdot\xi(r^1)]$ in (2) the following equation is obtained,

$$A_{sc}=f_0\sum_{r^1}\exp(iK\cdot r^1)[1+iK\cdot\xi(r^1)-\tfrac{1}{2}(K\cdot\xi(r^1))^2+\cdots]. \tag{3}$$

It is then shown that the first term in (3) corresponds to the Bragg reflection from the undisplaced lattice. The second term corresponds to the diffuse scattering in the neighborhood of reciprocal lattice points. The third term, which is coherent with the first, represents the diminution in the intensity of the Bragg reflection resulting from the presence of the defects in the crystal.

By substituting (1) for $\xi(r^1)$ in (3) the expressions for diffuse scattering, and the reduction in intensity of reflexions, are obtained for the case of a crystal containing several kinds of defect.

The distribution of diffuse scattering intensity in reciprocal space is considered. Two types of distributions are deduced. Interstitial defects in the body-centered position and substitutional defects give rise to a lemniscate type of distribution of diffuse scattered intensity about a lattice point. Interstitial defects in dodecahedral positions, and pairs of defects, show an ellipsoidal distribution.

W. Macintyre (Boulder, Colo.).

Groschwitz, E.; und Siebertz, K. Zur Frage der Plasmaschwingungen und -Wellen in Halbleitern. I. Z. Naturf. **11a** (1956), 482–491.

Due to the close relationship between an ionized gas and a semi-conductor, a study of plasma oscillations reveals many properties and structures of semiconductors in the presence of electric and magnetic fields. The purpose of this paper is to investigate the plasma oscillations from which one could obtain the mascoscopic properties of these materials. Here the analysis is carried along classical lines. By making the following assumptions about the nature of the plasma, namely: (a) the effective masses of the conducting electrons and the defect electrons are isotropic, (b) the influence of the fields on the velocity distribution of the conducting electrons and holes are neglected and the damping constant of both charge carriers is isotropic and independent of the energy of the particles, and the static field is so small that the electron temperature is effectively the same as the crystal temperature, (c) the magnetic properties of the semi-conductor are neglected by assuming the permeability to be one, the authors have derived the field equations and the equations of motion for a non-stationary inhomogeneous plasma. By limiting the case to periodic systems, the field equations, continuity equation and the equations of motion are given for an exponential type perturbation field. The authors have treated in detail a simplified type of plasma oscillations due to a homogeneous one-dimensional model semi-conductor with one kind of charge carriers (electrons) under the influence of a periodic field. Expressions for the conductivity, susceptibility and frequency of damped and undamped oscillations are derived and the results are shown in a number of curves. Finally, a special case of one-dimensional unbounded homogeneous and space free charge semi-conductor has been explicitly worked out, provided the electron concentration remains constant in space and time throughout the semi-conductor. *N. Chako* (Flushing, N.Y.).

Huber, A. Eine Näherungslösung für das Erstarrungsproblem. Z. Angew. Math. Mech. **36** (1956), 301.

Dans l'étude du problème de solidification lorsque les conditions empiriques ne sont pas suffisamment connues la recherche d'une solution exacte nécessite des calculs assez difficiles; il est quelquefois préférable de rechercher une solution approximative qui rend bien compte de l'évolution du phénomène.

L'auteur montre par un exemple particulier (la formation d'une goutte sphérique) comment il faut opérer pour la recherche de cette solution approximative.

M. Kiveliovitch (Paris).

Elliott, J. P.; Judd, B. R.; and Runciman, W. A. Energy levels in rare-earth ions. Proc. Roy. Soc. London. Ser. A. **240** (1957), 509–523.

The rare-earth ions are of importance in the theory of magnetism because of their occurrence in magnetic salts. This paper supplies the foundation for the use of the elegant group-theoretical methods of Racah in calculating the free-ion energy levels and the effect of the solid upon them. In such complex configurations these methods are nearly indispensable. *P. W. Anderson.*

Miasek, Maria. Tight-binding method for hexagonal close-packed structure. Phys. Rev. (2) **107** (1957), 92–95.

Tables for reduction of the matrix components of energy in the tight-binding approximation have been prepared for the hexagonal close-packed lattice.

Author's summary.

See also: Functions of Real Variables: Rosenstock. **Statistical Thermodynamics and Mechanics:** Bordoni. **Optics, Electromagnetic Theory, Circuits:** Cowley and Moodie.

Fluid Mechanics, Acoustics

Banerji, S. K. On the motion of iso-surfaces in a fluid medium. Bull. Calcutta Math. Soc. **48** (1956), 1–8.

L'auteur étudie les mouvements des iso-surfaces (isobarique, isothermique et autres) en considérant le déplacement de la surface le long de la normale à la surface. On trouve de cette façon, pour des intervalles assez courts, le déplacement S suivant la normale. L'auteur étend ses calculs à des surfaces de discontinuité et à des mouvements des systèmes déformables.

M. Kiveliovitch (Paris).

★ **Седов, Л. И. [Sedov, L. I.] Плоские задачи гидродинамики и аэродинамики [Plane problems of hydrodynamics and aerodynamics.]** Gosudarstv. Izdat. Tehn. -Teor. Lit., Moscow-Leningrad, 1950. 443 pp. 20.40 rubles.

This is a revision and expansion of the author's similarly entitled monograph [Uspehi Mat. Nauk **6** (1939), 120–182; MR **1**, 89]. The first seven chapters, comprising about three-quarters of the book, are devoted to applications of complex variable methods to problems of incompressible flow. The chapter headings are the following: I. Motion of an airofoil with constant circulation; II. Theory of thin wings; III. Theory of grids; IV. Shocks in incompressible flow; V. Theory of jets; VI. Flow in doubly connected regions (biplane problems); VII. Planing motion. Numerous special problems are worked out at length. General forms for forces, moments, inertia coefficients, etc. are developed in detail for airofoils in unsteady general motion, for grids, etc. Chapter II includes a lengthy discussion of harmonically oscillating airofoils and a generalization to linearized subsonic compressible flow. Chapter IV concerns impulsively started motions of floating cylinders or similar problems. Compressible flow is discussed in the last quarter of the book in the following chapters: VIII. General theory of steady motion of a gas; IX. Steady potential motion of a gas; X. Gas jets. While there are discussions of ordinary linearized compressible flow theory, the general tone is set by consideration of such topics as generalized Prandtl-Meyer flow; non-linearized examples of flows with vorticity; shocks, flame fronts, and condensation; and Chaplygin's method for gas jets. The material selected provides an excellent survey of Russian work in theoretical hydro- and aerodynamics, especially from 1930 to 1950. The book is frequently quoted by Russian authors and its wealth of examples and suggestive explanations must have provided stimulation and inspiration to numerous Russian students of fluid mechanics. *J. H. Giese.*

Sanoyan, V. G. Representation of certain plane and axial-symmetric flows by Fourier and Fourier-Bessel integrals. Akad. Nauk Armyan. SSR. Dokl. **22** (1956), no. 4, 149–156. (Russian. Armenian summary)

Author solves problems of determining velocity and pressure, and of constructing channels when their input velocity distribution is given, in the plane and axisymmetric incompressible flow case. Starting from Fourier integral representation of the velocity potential, he defines Fourier coefficients so that the longitudinal input velocity would have a given distribution. Solution obtained after certain manipulations is inadequate since boundary conditions cannot be satisfied. One other transformation corrects this inadequacy and solutions are used to solve certain problems: velocity distribution in the transverse section of a channel with the shape of a hyperbola; shape of plane channels for given values of the velocity and stream functions. These problems are solved for the plane and axisymmetric cases. *M. D. Friedman.*

Weber, J. The calculation of the pressure distribution over the surface of two-dimensional and swept wings with symmetrical aerofoil sections. Aero. Res. Council, Rep. and Memo. no. **2918** (1953), 63 pp. (1956).

Die Berechnung der Druckverteilung auf der Berandung eines Tragflügelprofils in einer ebenen stationären Strömung einer inkompressiblen reibungslosen Flüssigkeit ist in den letzten Jahrzehnten so ausgiebig behandelt worden, dass man sich die Aufgabe stellen kann, die zweckmäßigste Methode auszuwählen, die bis auf weiteres nicht übertroffen werden dürfte. Das Ziel des Verf. ist, darüber hinaus eine Näherungsmethode zu finden, die bei erträglichem Rechenaufwand den Fehler in tragbaren Grenzen hält. — Die Grundlage bilden die Arbeiten von Riegels und Wittich [Jbuch. Deutsch. Luftfahrtforschung **1942**, I 120–I 132], Riegels [Ing.-Arch. **16** (1948), 373–376; **17** (1949), 94–106; **18** (1950), 329; MR **10**, 490; **11**, 274; **12**, 646], ferner von Goldstein [Aero. Res. Council Rep. no. **5804** (1942); J. Aero. Sci. **15** (1948), 189–220], Twaites [Aero. Res. Council, Rep. and Memo. no. **2166** (1945)], Watson [ibid. no. **2176** (1945); MR **9**, 211]. Unter Beschränkung auf symmetrische Profile werden die Anströmung parallel und senkrecht zur Flügelsehne getrennt betrachtet und Näherungsformeln für die Geschwindigkeitsverteilung auf der Kontur gefunden, die einerseits auf konforme Abbildung verzichten, andererseits aber doch für dicke Profile brauchbar sind, wie ein Vergleich mit den exakten Rechnungen für das elliptische und das Joukowski-Profil zeigt. Es folgen Erweiterungen auf den schiebenden und auf den Mittelschnitt des gepfeilten unendlichen Flügels und schließlich numerische Methoden zur Berechnung einiger in den Formeln auftretenden singulären Integrale. *K. Maruhn* (Dresden).

Weber, J. The calculation of the pressure distribution on thick wings of small aspect ratio at zero lift in subsonic flow. Aero. Res. Council, Rep. and Memo. no. **2993** (1954), 45 pp. (1957).

The velocity distribution according to linearized theory is calculated by integration over the source distribution in the chord plane. For the particular planforms and spanwise distributions of thickness considered, the spanwise integration can be performed explicitly; for the chordwise integration the method of mechanical quadrature of Gauss is used. This effectively extends to these wings the techniques previously developed for two-dimensional and swept airfoils [Kücheman and Weber, same Rep. no. **2908** (1953); see also the paper reviewed above] by which the velocity distribution is computed directly in terms of the section ordinates. The cases treated in detail here are only the center sections of (i) rectangular wings with linearly varying thickness/chord ratio, (ii) rectangular wings with linearly varying thickness/chord ratio, and (iii) untapered wings of 45° and 60° sweepback, of constant section. Tables of coefficients are given. All the calculations are carried out for incompressible flow, the intention being that the correction for compressibility will be made by standard methods.

The assumptions of linearized theory and their effects on the velocity distribution are discussed, and an improved approximation for the leading-edge region is suggested. Next, the slender-body approximation and its application to wings is reviewed. The approximate theory of Keune [Roy. Inst. Tech., Div. Aero., Stockholm, KTH-Aero. Tech. Note no. **21** (1952)], which introduces certain slender-body approximations in linearized theory, is discussed. Finally, the velocity distributions on ellipsoids with three unequal axes are calculated by slender-body theory, Keune's theory, and the present method (linearized theory), and the results are compared with exact results. *W. R. Sears* (Ithaca, N.Y.).

Preston, J. H. Note on the circulation in circuits which cut the streamlines in the wake of an aerofoil at right-angles. Aero. Res. Council, Rep. and Memo. no. 2957 (1954), 8 pp. (1956).

Re-examination of a theorem on the title-problem, previously investigated by G. I. Taylor (same Rep.& Memo. no. 989, 1924), the author (no. 1996, 1943; no. 2725, 1949), G. Temple (unpublished) and D. A. Spence (J. Aeronaut. Sci., **21** (1954), 577–587, 620; MR **16**, 189).

Nickel, K. Ein Sonderfall des senkrechten Profilgitters bei beliebigen instationären Bewegungen. Ing.-Arch. **25** (1957), 134–139.

In einer früheren Arbeit des Verf. [Ing.-Arch. **23** (1955), 179–188; MR **17**, 98] wurde die ebene instationäre Bewegung einer volumenbeständigen, reibungsfreien Flüssigkeit durch ein senkrechtes Flügelgitter behandelt. In der vorliegenden Arbeit wird der Grenzfall unendlich tiefer Gitterschaufeln betrachtet, der einem Gitter mit endlichen Schaufeln entspricht, wenn die instationäre Bewegung erst kurze Zeit andauert. Hierbei vereinfachen sich gegenüber dem allgemeinen Fall die Integralgleichungen des Problems und die Lösung ganz erheblich. Man erhält eine geschlossene Integraldarstellung für die Auftriebsverteilung über die Flügeltiefe, in der bemerkenswerterweise die Stärke der freien Wirbel nicht vorkommt. Für letztere erhält man eine Abelsche Integralgleichung, die sich ja ebenfalls explizit lösen lässt. Neben dem Gitter mit unendlicher Tiefe und endlichem Abstand der Schaufeln wird noch der Fall unendlichen Abstands (Einzelprofil) mit behandelt. Eine Reihe charakteristischer Beispiele zeigt, dass eine Übertragung auf das allgemeine Gitter im allgemeinen nicht zulässig ist. *K. Maruhn* (Dresden).

Dumitrescu, D. Une solution exacte du système d'équations différentielles Navier-Stokes, pour les fluides pesants à surface libre. Com. Acad. R. P. Române **4** (1954), 115–122. (Romanian. Russian and French summaries)

The author considers the steady rectilinear flow of a heavy viscous incompressible fluid with a free surface. After carrying out the known reduction of the equations of motion to a boundary problem for Poisson's equation in the plane with constant right-hand member, and then to Dirichlet's problem, the author investigates in two cases the law for the velocity distribution in the presence of a plane bottom making some angle with the horizontal: 1) the region is infinitely wide; 2) the region of flow is a rectangular canal of constant finite section.

The boundary conditions are given in the following form: in the first case the velocity vanishes on the bottom and on the free surface has a given constant value; in the second case the velocity vanishes on the bottom and sides and on the free surface follows a parabolic law. Such boundary conditions on the free surface are not consistent with those accepted in the dynamics of a viscous fluid. [See Kochin, Kibel, and Roze, Theoretical hydromechanics, vol. II, Chap. 2, §§6, 11, Gostehizdat, Moscow, 1948; cf. MR **17**, 307.] There are also misprints.

D. E. Dolidze (RŽ Meh **1955**, no. 1883).

Kočina, I. N. On waves on the surface of separation of two fluids flowing at an angle to each other. Prikl. Mat. Meh. **19** (1955), 628–634. (Russian)

Let a fluid of density ρ_1 fill the region $z<0$ and flow in the direction OX with velocity c_1. A second fluid of density $\rho_2<\rho_1$, fills the region $z>0$ and flows with velocity c_2 in a direction making an angle α with OX. In addition a stationary body S is assumed to be present in the lower fluid. Assuming linearized boundary conditions at the interface, the author finds velocity potentials in the two fluids corresponding to a source in the lower fluid, then finds the velocity potentials for the problem stated above in terms of Kočin's H-function. (This requires solution of an integral equation for density of sources over S). The asymptotic form of the interface at large distances from S is found. The form depends upon α and can be markedly different from the usual ship-wave pattern. The program is then repeated when the lower fluid is bounded below by a horizontal bottom. The exposition suppresses a considerable amount of work. *J. V. Wehausen.*

★ **Czwalina, Arthur. Die Mechanik des schwimmenden Körpers.** Akademische Verlagsgesellschaft, Geest & Portig K.-G., Leipzig, 1956. 129 pp. DM 12.00.

The author has computed the positions of stable equilibrium for a floating body for a substantial number of geometrical shapes (for example, box, cylinder, cone, segment of a sphere, octahedron). In addition, he has computed the periods for small heaving and pitching motion of the bodies about their various equilibrium positions (in keeping with the character of the work only hydrostatic forces are included in this computation). The introduction indicates that the author regards the book as a continuation of work begun by Archimedes. *J. V. Wehausen.*

Moiseev, N. N. The problem of small oscillations of an open vessel with fluid acted upon by an elastic force. Ukrain. Mat. Ž. **4** (1952), 168–173. (Russian)

Consider a rectangular vessel of mass m_2 partially filled with fluid of mass m_1 and let the vessel be movable and subject to a restoring force proportional to the displacement about a position of equilibrium. The author determines the oscillations of the fluid, assuming infinitesimal waves, and the characteristic frequencies of the system. *J. V. Wehausen* (Berkeley, Calif.).

★ **Brillouet, Georges. Étude de quelques problèmes sur les ondes liquides de gravité.** Publ. Sci. Tech. Ministère de l'Air, Paris, no. 329, 1957. ix+145 pp. 1700 francs.

The linearized plane problem of water waves on a sloping beach consists in finding a solution $\phi(x, y)$ of Laplace's equation $\nabla^2\phi=0$ in the sector $-\alpha\leqq\phi\leqq 0$, subject to the boundary conditions:

$$\frac{\partial\phi}{\partial n}-\phi=0 \text{ on } \theta=0,$$

$$\frac{\partial\phi}{\partial n}=0 \text{ on } \theta=\alpha,$$

where $\partial/\partial n$ denotes normal differentiation out of the region. This problem has been treated by Stoker and Lewy for angles $\alpha=\pi/2q$ and $\alpha=p\pi/2q$ respectively, where p and q are integers without common factors. [Their results have since been extended to arbitrary angles by Isaacson and by Roseau, and to the equation $(\nabla^2-k^2)\phi=0$ by Peters and by Roseau; for references see J. J. Stoker, Water waves, Interscience, New York, 1957.] In the present tract, the author has made a new study of the problem for $\alpha=p\pi/2q$, paying special attention to motivation and uniqueness. Incidentally, he obtains some notable simplifications. There are some brief remarks on the analogous problem for $(\nabla^2-k^2)\phi=0$. This work will make the work of Stoker and of Lewy accessible to a wider class of readers. *F. Ursell* (Cambridge, Mass.).

Dombrovskiĭ, G. A. On subsonic jet flow about a grid of plane plates. Dokl. Akad. Nauk SSSR (N.S.) **111** (1956), 312–315. (Russian)

The author has previously described a method for approximately integrating the equations of subsonic plane compressible flow [same Dokl. (N.S.) **103** (1955), 31–34; MR **17**, 681]. Now he applies it to subsonic flow with Helmholtz wakes behind a periodic grid of flat plates. *J. Giese* (Aberdeen, Md.).

Garabedian, P. R. Calculation of axially symmetric cavities and jets. Pacific J. Math. **6** (1956), 611–684.

The author continues his brilliant attack on the difficult problem of effectively computing axially symmetric free boundary flows. His present approach is based on expanding all quantities in powers of $\delta=p/(p+2)$, where $p+2$ is the number of space dimensions involved. Among the many analytical arguments used to support the calculations, may be mentioned the exact formula $4hD=3(QV-M)$ for the cavity drag D of the Riabouchinsky flow with cavity length $2h$ past a cone. Here Q is the cavitation number, V the cavity plus obstacle volume, and M the virtual mass. The free boundary depends monotonely on the dimension number; for fixed point of flow separation, as Q varies, we have $C_D(Q)=(1+Q)C_D(0)+o(Q)$.

The final conclusions are: (i) the contraction coefficient of the jet through a circular hole is 0.58, instead of the accepted value of 0.61, and (ii) the C_D of a disc is about 0.83, instead of 0.81 as often quoted. No rigorous error analysis is included, and the sensitivity of the disc calculations to Q is disturbing, but the analysis certainly goes beyond that of previous authors. *G. Birkhoff.*

Curtet, Roger. Contribution à l'étude théorique du mélange des jets de révolution. C. R. Acad. Sci. Paris **244** (1957), 1450–1453.

Dans une note précédente [mêmes C. R. **241** (1955), 621–622] M. Craya et l'Auteur ont établi une théorie approchée du mélange des jets en espace confiné, pour un écoulement plan en fluide incompressible.

L'Auteur présente ici quelques aspects de la théorie étendue au problème des jets de révolution, mettant en évidence une étroite analogie entre ce problème et le problème plan. *R. Gerber* (Toulon).

Keller, Joseph B. Teapot effect. J. Appl. Phys. **28** (1957), 859–864.

Tea poured from a teapot often runs along the under side of the spout instead of falling directly into the cup. Recent experiments have shown that this effect is due neither to surface tension nor to adhesion. The author's explanation is this. When the liquid rounds the lip the speed is greatest there and so the atmospheric pressure pushes the liquid against the lip. This enables the flow to turn the corner and then to continue along the under side of the spout because the surrounding air supports it. The author uses classical hydrodynamics to examine the flow through a two-dimensional spout and finds a new flow bounded by a single free streamline which separates from the underside of the upper face of the spout and continues round the edge of the lower face and along the underside to which it eventually (in the absence of gravity) becomes parallel. This upside-down flow, although unstable, can continue far enough up the underface to explain the phenomenon. *L. M. Milne-Thomson.*

Avališvili, L. E. Non-stationary boundary problem of Oseen. Soobšč. Akad. Nauk Gruzin. SSR **17** (1956), 489–494. (Russian)

★ **Avalishvili, L. E. Nonstationary boundary problem of Oseen.** Translated by Morris D. Friedman, 572 California St., Newtonville 60, Mass., 1957. 6 pp.

Translation of the above article.

Kiselev, A. A. Unsteady flow of a viscous incompressible fluid in a bounded three-dimensional region. Translated by Morris D. Friedman, 572 California St., Newtonville 60, Mass., 1956. 5 pp.

Translation of Dokl. Akad. Nauk SSSR N.S. **106** (1956), no. 1, 27–30.

Lakshmana Rao, S. K. Harmonic analysis of the spatial flow of an incompressible viscous fluid. Proc. Indian Acad. Sci. Sect. A. **44** (1956), 6–14.

The author extends to three-dimensional space the method introduced by the reviewer [Quart. Appl. Math. **6** (1948), 1–13; MR **9**, 631] for two-dimensional flow of an incompressible viscous fluid; the Fourier transforms of the Navier-Stokes equations are given, first when the viscous fluid is contained in a bounded domain with velocity vanishing at the boundary and next when the

flow region is the entire space, the velocity vanishing sufficiently strongly at infinity. Bounds are given for the spectral function of the kinetic energy of the flow. The integro-differential equations for the components of the velocity are integrated in the particular case when the non-linear terms are negligible. *J. Kampé de Fériet.*

★**Слезкин, Н. А. [Slëzkin, N. A.] Динамика вязкой несжимаемой жидкости. [Dynamics of viscous incompressible fluid.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1955. 520 pp. 10.90 rubles.

This is a rather comprehensive exposition of the theory of incompressible viscous fluids. The chapter headings give a good idea of the organization and topics. I. Rate of strain. Stress components. II. Differential equations of motion of a viscous fluid. III. General properties of motion of a viscous fluid. IV. Exact solutions for steady motion. V. Motion for small Reynold's numbers — Stokes' method. VI. Lubrication theory. VII. Motion for small Reynolds numbers — Oseen's method. VIII. Boundary-layer theory. IX. Unsteady motion. X. Development of laminar motion (i.e. at inlets to pipes and diffusors). XI. Stability of laminar flows. XII. Turbulent motion. An introduction (pp. 9–24) provides a fairly comprehensive historical sketch. As might be expected in a general exposition, the last chapter can give only an introduction to the various aspects of turbulent motion. The author frequently gives references to further development of material treated in the text. The book reads easily. *J. V. Wehausen.*

Iacob, Caius. L'effort exercé sur une paroi en contact avec un fluide visqueux. Rev. Univ. "C. I. Parhon" Politehn. Bucureşti. Ser. Şti. Nat. **3** (1954), no. 4–5, 133–138. (Romanian. Russian and French summaries)

R. Berker [C. R. Acad. Sci. Paris **232** (1951), 148–149; MR **12**, 551] a obtenue une expression de l'effort élémentaire exercé sur une paroi fixe en contact avec un fluide visqueux par un procédé utilisant certaines formules intégrales. Dans le présent travail on considère le cas ou la paroi solide est modile et l'on donne l'expression de l'effort correspondant sous la forme

$$\mathbf{T}=[p-(\lambda+\mu)\theta]\mathbf{n}-\mu\,\frac{\partial \mathbf{v}}{\partial n}+\mu\boldsymbol{\omega}\times\mathbf{n}$$

ou bien sous la forme

$$\mathbf{T}=[p-(\lambda+2\mu)\theta]\mathbf{n}+\mu\mathbf{n}\times\operatorname{rot}\mathbf{v}-2\mu\mathbf{n}\times\boldsymbol{\omega};$$

ces résultats sont établis sans avoir recours à des formules intégrales. (Résumé de l'auteur). *J. V. Wehausen.*

Carrier, G. F.; and DiPrima, R. C. On the unsteady motion of a viscous fluid past a semi-infinite flat plate. J. Math. Phys. **35** (1957), 359–383.

The velocity of the undisturbed stream is taken as $U_0+U_1\exp(i\omega t)$. The equations of motion are linearized and contain a real adjustable parameter c; they reduce to Oseen's equations if $c=1$, $U_1=0$. If $(\omega\nu)^{\frac{1}{2}}/U_0$ and $(\omega\nu)^{\frac{1}{2}}/U_1$ are both large, the interaction between the steady and the oscillatory parts of the motion can be ignored, and a formula for the skin friction is derived. In the general case, when there is interaction, asymptotic formulae are found for the skin friction near, and at a great distance from, the leading edge of the plate. *W. R. Dean.*

Mirels, Harold; and Braun, W. H. Nonuniformities in shock-tube flow due to unsteady-boundary-layer action. NACA Tech. Note no. **4021** (1957), 45 pp.

La couche limite le long des parois d'un tube à choc introduit dans un écoulement potentiel des gradients de pression et de vitesse; ces effets sont évalués pour une couche limite qui est soit entiérement laminaire, soit entièrement turbulente, et dont l'épaisseur est faible par rapport au diamètre du tube. Les auteurs admettent que l'action de la couche limite équivaut à une distribution rectiligne de sources et que le région dans lequelle l'écoulement est constitué par des ondes simples concourantes peut être négligée. Les variations de la pression et de la vitesse, dans le région chaude (comprise entre le choc et la surface de discontinuité) sont calculées et représentées sur de nombreux graphiques, pour un tube à choc air-air. *H. Cabannes* (Marseille).

Englert, Gerald W. Estimation of compressible boundary-layer growth over insulated surfaces with pressure gradient. NACA Tech. Note no. **4022** (1957), 35 pp.

This is an extension of Truckenbrodt's method, based on the kinetic-energy-integral equation [Ing. Arch. **20** (1952), 211–228=NACA Tech. Memo. no. **1379** (1955)], for compressible flow over insulated two-dimensional and axisymmetric surfaces, assuming Prandtl number equal to 1. The extension is carried out by means of the Stewartson-Illingworth transformation. Both laminar and turbulent boundary layers are considered. The resulting equations are applied to several cases of turbulent boundary-layer flow for which experimental data are available, including flows with adverse, zero, and favorable pressure gradients. Agreement with experimental data is found to be good. *W. R. Sears* (Ithaca, N.Y.).

Garabedian, P. R. On steady-state bubbles generated by Taylor instability. Proc. Roy. Soc. London. Ser. A. **241** (1957), 423–431.

A theory of the steady motion of a long bubble rising through an infinite plane vertical tube of liquid is presented. It is shown that, contrary to current belief, the flow is not uniquely determined by the width h of the tube and the acceleration of gravity g alone, but that the speed U of the bubble can also be prescribed. However, a criterion of stability singles out the unique physically significant flow of this type as the one which maximizes the velocity of the bubble. With the aid of a difference-differential equation derived from the free-boundary condition, the Froude number $U/(gh)^{\frac{1}{2}}$ for this latter case is estimated to exceed 0.2363, a value slightly higher than that indicated by earlier work on the problem. (Author's summary) *D. W. Dunn* (Ottawa, Ont.).

Smith, A. M. O. On the growth of Taylor-Görtler vortices along highly concave walls. Quart. Appl. Math. **13**, 233–262 (1955).

The primary object of this study has been to prepare a chart of computing the growth of Taylor-Görtler vortices in laminar flow along walls of both high and low concave curvature. Taylor-Görtler vortices are streamwise vortices having alternate right- and lefthand rotation that may develop in the laminar boundary layer along a concave surface. The equations of motion are derived anew and re-examined with regard to the importance of the various terms. The final equations used in preparation of the chart are found to be valid for radii of curvature as small as 30 times the boundary layer thickness. Furthermore it

is shown that the equations are not restricted in validity to cases of constant wall curvature, constant free stream velocities, or to boundary layers of constant thickness. Whereas the previous analysis of Taylor and Görtler assumed the vortex to grow exponentially as a function of time, the present study recasts the growth into a more convenient form in which the vortex grows as a function of the distance. The solution is an eigenvalue problem, which in the present study has been solved mainly by Galerkin's method — a variational method. Both the eigenvalues and the eigenfunctions are presented, the former in the aforementioned chart. It is possible to compare the solutions for neutral stability with those given by Görtler. The two solutions are in approximate agreement. A second method of solution is also described. This method is believed to offer considerable improvement, provided a high speed digital computer is available. In the one case checked by both methods agreement was within 2%. Finally, the stability chart was applied to all the known experimental data concerning the effect of concave curvature on the transition point. The well-known parameter $R_\theta(\theta/r)^{\frac{1}{2}}$ is shown to be inadequate as a indicator of the transition point. Instead the experimental data indicate that an apparent amplification factor, $\exp \int \beta dx$, is a much better measure. Available results show that transition of this type will occur when $\int \beta dx$ reaches a value of about ten. (Author's abstract.)

R. M. Morris (Cardiff).

★ **Hopf, Eberhard. On the application of functional calculus to the statistical theory of turbulence.** Applied probability. Proceedings of Symposia in Applied Mathematics, Vol. VII, pp. 41–50. McGraw-Hill Book Co., New York-Toronto-London, for the American Mathematical Society, Providence, R. I., 1957. $5.00.

The methods of functional calculus first applied to the theory of turbulence by the author [J. Rational Mech. Anal. **1** (1952), 87–123; MR **15**, 478] are here briefly reviewed. A complete statistical description of the velocity field is obtained by employing a characteristic functional Φ, the derivatives of which yield the velocity correlations of all orders. The principal difficulty of the subject arises when one attempts to solve the functional differential equation satisfied by Φ which follows from the Navier-Stokes equation. The only known non-trivial solution of the Φ-equation is time independent with zero viscosity [E. Hopf and E. W. Titt, ibid. **2** (1953), 587–591; MR **15**, 478]; for a Gaussian distribution, this solution leads to an energy distribution in k-space which is at variance with the one given by Kolmogoroff. Two approximate methods of solution are then discussed. In the first method, one assumes a Bruns-Charlier expansion

$$\Phi=\exp(\Phi_2)[1+\Phi_3+\cdots].$$

The first meaningful approximation is then obtained by neglecting Φ_4, thus leaving a pair of coupled equations for Φ_2 and Φ_3; this is equivalent to assuming the relation between fourth and second order moments valid for a normal distribution, in which form the theory has been fairly fully worked out [I. Proudman and W. H. Reid, Philos. Trans. Roy. Soc. London. Ser. A. **247** (1954), 163–189; MR **16**, 299; T. Tatsumi, Proc. Roy. Soc. London. Ser. A. **239** (1947), 16–45; MR **18**, 694]. In the second method, one attempts to obtain solutions by separating the variables in the Φ-equation; as yet this method has not proved very successful. There can be little doubt as to the importance of this method of attack on the turbulence problem, but there yet remain some very formidable difficulties to be overcome. *W. H. Reid.*

Tollmien, Walter. Spektralanalyse der homogenen Turbulenz. Z. Flugwiss. **4** (1956), 195–198.

Mostly expository paper with valuable historical remarks, giving frequently new insight into the problem considered; the main contribution of the author consists in a new form of the spectral tensor, in the case of an incompressible fluid; these derived formulas are of the general form given by the reviewer [C. R. Acad. Sci. Paris **227** (1948), 760–761; MR **10**, 200], but under the assumption that the Fourier-integrals, expressing the correlation tensor in terms of the spectral tensor, could be inverted. *J. Kampé de Fériet* (Lille).

Eggleston, John M.; and Diederich, Franklin W. Theoretical calculation of the power spectra of the rolling and yawing moments on a wing in random turbulence. NACA Tech. Note no. **3864** (1956), ii+56 pp.

The paper is concerned with the computation of the aerodynamic forces and moments produced by the lifting surfaces flying through turbulent air, when the gust velocities are functions of position and time known only in a statistical sense. In the first part a theoretical analysis is made defining the power spectra of the rolling and yawing moments of a wing in terms of the statistical characteristics of the gusts. The following assumptions are made: (a) the turbulence is homogeneous and isotropic; (b) time-correlations are equivalent to space-correlation along the path of the plane; (c) the indicial-response influence function for the rolling and yawing moments can be expressed as a product of a function of distance along the span only and a function of distance along the path (time) only; (d) the wing is a thin lifting surface of arbitrary plan form; it is rigid and performs small motions about a mean steady flight condition. The second part contains a brief discussion of the assumptions concerning the nature of turbulence and the aerodynamic forces and some indications about the applications of the results.

J. Kampé de Fériet (Lille).

Malkus, W. V. R. Outline of a theory of turbulent shear flow. J. Fluid Mech. **1** (1956), 521–539.

The author considers the fully three dimensional turbulent flow between 2 fixed parallel plates. A new stability condition is assumed: i.e. that the mean non-linear momentum transport terms in the equations of motion $\overline{v'\nabla v'}$ are entirely stabilizing and that the only energy source for those disturbances which enter into the mean momentum transport is the pressure field of the mean motion. A limiting statistical stability condition on the spatial structure of the mean momentum transport is found, which leads, for any mean velocity $v(z)$ satisfying $v''\leq 0$, from the Orr-Sommerfeld equation, to the scale of the smallest statistically unstable motion. A variational approach is devised to compute the momentum transport spectrum and the mean velocity profile which lead to a stable mean field of maximum dissipation rate; the mathematical analysis is rather laborious and considerable computational work is involved. *J. Kampé de Fériet.*

Smith, F. B. The diffusion of smoke from a continuous elevated point-source into a turbulent atmosphere. J. Fluid Mech. **2** (1957), 49–76.

The problem is to determine the downwind concentra-

tion of non-buoyant smoke emitted from a continuous elevated point-source in a turbulent air-stream; the author uses the classical diffusion equation, with the following assumptions: (a) the mean velocity is represented by: $u=u_0(z+h)^\alpha$ where z is the height relative to the source height h; (b) the diffusion in the direction of the main stream is neglected $K_x=0$; (c) the vertical and the cross-wind diffusion coefficients are equal: $K_z=K_y=K_0(z+h)^{-\alpha}$. For the momentum C_0 and C_2 of the concentration C, exact solutions are given for $\alpha=\frac{1}{2}$ and approximated solutions for $\alpha\neq\frac{1}{2}$, along lines lying in the cross-wind direction at ground level. Assuming that the cross-wind distribution is Gaussian, the concentration $C(y)$ is deduced from C_0 and C_2. *J. Kampé de Fériet.*

Craya, Antoine. Sur les équations de la turbulence homogène en présence d'un champ de vitesse moyenne. C. R. Acad. Sci. Paris **244** (1957), 847–849.

Craya, Antoine. Une méthode d'investigation des corrélations doubles pour la turbulence homogène en présence de vitesses moyennes. C. R. Acad. Sci. Paris **244** (1957), 1448–1450.

Craya, Antoine. Sur la structure des corrélations triples en turbulence homogène associée à un champ de vitesses moyennes. C. R. Acad. Sci. Paris **244** (1957), 1609–1611.

The author considers a spatially homogeneous velocity field superposed on a mean stream such that $\partial u_i/\partial x_k=\lambda_{ik}=$const; previously the particular case in which only $\lambda_{12}\neq0$ has been treated by J. M. Burgers and M. Mitchner [Nederl. Akad. Wetensch. Proc. Ser. A. **56** (1953), 228–235, 343–354; MR **15**, 573]. The 4 spectral tensors $\Phi_{i,j}$, $\Theta_{i,j,k}$, $\Phi_{i,j,k}$ and $\Theta_{i,j,k,l}$ corresponding respectively to the double, triple (at 2 and 3 points) and the quadruple (at 3 points) correlation tensors, are introduced and the differential equations for the evolution of the spectrum are given in terms of these tensors. Supposing the turbulence stationary in t, the first of these equations, concerning the double correlation, is considered and a method of approach, using special axes in the wave-numbers space, is devised; next, account is taken of the triple correlation; it is asserted that the general formula for the $\Phi_{i,j}$ (not explicitly given here) could be derived from these equations. *J. Kampé de Fériet* (Lille).

Ellison, T. H. Turbulent transport of heat and momentum from an infinite rough plane. J. Fluid Mech. **2** (1957), 456–466.

This paper considers the turbulent motion in the region of essentially constant shear stress and vertical heat transport that lies deep within a thick boundary layer on a heated, horizontal plane. In the first part, it is assumed that the motion is uniquely determined, except for a velocity of translation, by the shear stress and the heat flux and the dimensional consequences of this result are developed. The second part uses the equations for the turbulent energy, the heat transport and the mean square density fluctuation to obtain a relation between the transport coefficients, the flux Richardson number and some ratios of the component turbulent intensities and of characteristic decay times. Vertical transport by the turbulence of the quantities concerned is assumed to be negligible.

The problem is of great importance for meteorology and is considered in this context. The principal results of the dimensional analysis are that the mean velocity profile can take only two forms, depending on whether the mean density gradient is stable or unstable, and that the eddy viscosity is independent of height in very stable conditions. The second part of the analysis indicates a critical value of 0.15 for the flux Richardson number but no upper limit for the ordinary Richardson number. A number of other conclusions are described and discussed. *A. A. Townsend* (Cambridge, England).

Cabannes, Henri. Sur les mouvements rectilignes d'un fluide compressible. I, II. Rech. Aéro. no. **59** (1957), 3–11.

I. A general formulation of the one dimensional, non-isentropic flow of a compressible fluid without viscosity and heat conduction is given in terms of a trajectory function ψ and the time t. This is reduced to a second order equation in a new flow variable and a general solution is indicated which is dependent on only two arbitrary functions. A particular solution of the same problem, first given by C. W. Jones [Proc. Roy. Soc. London. Ser. A. **228** (1955), 82–99; MR **16**, 641] is obtained as preparation for its application to part II of this paper.

II. The work in part II appears to be the same as in C.R. Acad. Sci. Paris **243** (1956), 1482–1484 [MR **18**, 696]. The flow problem described above is reconsidered with addition of viscosity but not heat condition. The same variables are used as above. The resulting equations are treated by developing the solutions in terms of a power series expansion in a small parameter. The system of equations produced may then be successively integrated. When the coefficient of heat conduction is not zero, these methods do not apply. For this more complicated case, an extension of the particular solution of Jones given in I is used. *Hirsh Cohen* (Troy, N.Y.).

Legendre, Robert. Ecoulement subsonique transversal à un secteur angulaire plan. C. R. Acad. Sci. Paris **243** (1956), 1716–1718.

The object of this note is to calculate the degree of homogeneity of the velocity potential for a subsonic flow over an infinite plane sector. The results obtained serve to extend those previously obtained by P. Germain [Rech. Aéro. no. **44** (1955), 3–8; MR **16**, 969]. *R. C. F. Bartels* (Ann Arbor, Mich.).

Yeh, Hsuan. Sears function in unsteady flows. J. Aero. Sci. **24** (1957), 546–547.

The real and imaginary parts of Sears' function [same J. **8** (1941), 104–108; MR **3**, 94], $S(w)$, are tabulated to five figures for $w=1(1)50$ and plotted in a polar diagram for $w=15(1)50$. An asymptotic formula, satisfactory for $w>50$, also is given. *John W. Miles* (Los Angeles).

Lighthill, M. J. Contributions to the theory of the Pitot-tube displacement effect. J. Fluid Mech. **2** (1957), 493–512.

In two previous papers, on "drift" [same J. **1** (1956), 31–53; **2** (1957), 311–312; MR **18**, 437, 965] and on the image of a vortex element in a rigid sphere [Proc. Cambridge Philos. Soc. **52** (1956), 317–321; MR **17**, 1249], the author has prepared the way for a study of the so-called "displacement effect" of pitot tubes in shear flow. This is the tendency of pitot tubes to measure the stagnation pressure, not at the same position in undisturbed flow, but at a displaced location, due to distortion of the stagnation streamline. In an introduction, the author dis-

cusses the history of the subject and its relation to the general field of hydrodynamics known as "secondary flows." He points out the inadequacy of two-dimensional studies of the phenomenon. He then proceeds to determine approximately the induced flow ahead of a spherical obstacle in an inviscid shear flow, using results of the papers mentioned above. This permits calculation, to first order, of the displacement of the stagnation streamline.

The author then proceeds to a rather detailed comparison with the theory of Hall [J. Fluid Mech. **1** (1956), 142–162; MR **18**, 528], which involved an important simplifying assumption. He concludes that Hall's assumption is a good one, and that its success is related to the fact that the image system of upstream vorticity contributes the largest part of the displacement effect; trailing vorticity downstream is not important. Finally, he discusses at some length the relevance of the sphere problem to the pitot-tube problem. In this discussion he uses some results of a forthcoming paper. The discussion touches qualitatively on the observation that the displacement effect is much reduced with sharp-lipped tubes and also in supersonic flow. *W. R. Sears.*

Žigulev, V. N. On a class of plane axisymmetric approximately sonic flows. Prikl. Mat. Meh. **20** (1956), 613–620. (Russian)

Černyĭ, G. G. Vortical flows of compressible gas in channels. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk 1956, no. 6, 55–62. (Russian)

Mangler, K. W. Ein Äquivalenzsatz für laminare Grenzschichten bei Hyperschallströmung. Z. Angew. Math. Mech. Sonderheft (**1956**), S22–S25.

In boundary layer theory for high speed flow the condition of zero velocity at the wall is replaced by the slip condition $V=L\partial V/\partial Z$, where L is related to the mean free path, V is the speed along the wall and Z is distance from the wall. In this paper it is shown that finding the flow in such a slip stream is equivalent to finding the flow in a boundary layer into which fluid is being injected or from which fluid is sucked. The equivalence theorem is based on the transformation $x=\bar{x}$, $y=\bar{y}$, $z=\bar{z}-\bar{g}(\bar{x},\bar{y})$, $\bar{w}=w-ug_x-vg_y$, which leaves the boundary layer equations of motion invariant. It is then shown that the thickness of the layer and the heat transfer is less for the flow with the slip condition than for the flow with a fixed boundary. Also the shearing stresses are larger where the pressure gradient is positive and smaller for negative pressure gradient. The two-dimensional case was carried out by Nonweiler [Coll. Aero. Cranfield Rep. no. 62 (1952); MR **14**, 1034]. *C. S. Morawetz* (New York, N.Y.).

Lighthill, M. J. Dynamics of a dissociating gas. I. Equilibrium flow. J. Fluid Mech. **2** (1957), 1–32.

This is a lucid introduction to the effects of dissociation in gas dynamics. The problem in view is that of air flow past a bluff body at speeds somewhat above 2 km/sec. Thermodynamic equilibrium is assumed; theories of near equilibrium for transport properties and of large departures from equilibrium being promised in Parts II and III.

Following a survey of the equilibrium statistical thermodynamics of a pure dissociating diatomic gas, a new model is introduced. This "ideal dissociating gas" is characterized by only three constants, the characteristic temperature, density and internal energy for dissociation. Physically, it may be regarded as having its vibrational modes always just half excited (so that at low temperatures the ratio of specific heats approaches 4/3 rather then 7/5). Thermodynamic properties of the ideal gas are derived, and the oblique shock wave relations deduced in the "strong-shock" approximation (including an elegant relation between the principal curvatures of any bow shock and the subsequent vorticity). Useful relations are given for the isentropic changes that take place along streamlines between shocks.

Various of these results are applied to the problem typified by a sphere flying at high Mach number. The Newtonian impact theory and its empirical modification are dismissed as lacking theoretical basis, in favor of the "Newtonian-plus-centrifugal" theory, which is the proper limit for large values of both Mach number and density ratio across the shock. It is suggested that the zero surface pressure sometimes predicted by the latter theory (which leads to singularities in higher approximations) corresponds to separation not of the flow but of the shock wave from the surface. An estimate is given for the subsequent shape of the shock. Finally, another approximation is applied to the region near the stagnation streamline. The fluid is assumed incompressible, but rotational in accord with the shock relations; and it is shown that a spherical shock corresponds to a concentric spherical body. The resulting surface pressure is within 1 per cent of that predicted by Freeman's second approximation based on the Newtonian-plus-centrifugal solution [same J. **1** (1956), 366–387; MR **18**, 534]. *M. D. Van Dyke.*

Todeschini, Bartolomeo. Correnti ipersoniche che prolungano una corrente uniforme. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. **90** (1956), 467–472.

It is well known that the transition from uniform to non-uniform steady supersonic plane flow occurs via a simple wave. The usual proof is based upon consideration of the map in the hodograph plane. The author gives a proof directly in the physical plane. He expands the velocity potential in the form $\Phi=\tilde{u}x+\sum \varepsilon^n c_n(x,y)$, where $\tilde{u}$ is the undisturbed velocity and ε a parameter. He writes the successive differential equations for c_n and confirms that the boundary condition at the junction with uniform flow forces c_n takes the form required for a simple wave. *J. Giese* (Aberdeen, Md.).

★ **Dorfner, Karl-Richard. Dreidimensionale Überschallprobleme der Gasdynamik.** Springer-Verlag, Berlin-Göttingen-Heidelberg, 1957. vii+150 pp. DM 26.00.

In spite of its somewhat broader title, the scope of this book is actually limited to the linearized theory of thin wings in steady flow at supersonic speeds. After an introductory chapter, which includes a section on discontinuous functions, the treatment is divided into three main chapters, concerned with singularity distributions, conical fields, and homogeneous flows, respectively. A feature of the book seems to be the systematic presentation of the various "thickness" and "lifting" problems whose solutions fall into these several categories. The mathematical treatment is largely conventional, utilizing the "finite part" of divergent integrals and similar devices of Cauchy-Hadamard theory. *W. R. Sears.*

Kuo, Y. H. Dissociation effects in hypersonic viscous flows. J. Aero. Sci. **24** (1957), 345–350.

The problem of boundary-layer flow is formulated for a gaseous mixture, resulting from the dissociation of a single perfect diatomic gas. *From the author's summary.*

Il'yušin, A. A. The law of plane sections in the aerodynamics of high supersonic velocities. Prikl. Mat. Meh. **20** (1956), 733–755. (Russian)

The paper contains a complete development of hypersonic small disturbance theory which can be compared with the alternative exposition by Van Dyke [NACA Rep. no. **1194** (1954); MR **17**, 209]. Application is made to the linearized theory of lifting surfaces, cones and ogives and cylindrical ducts. A higher order treatment of lifting surface theory is also given and the paper ends with a discussion of experimental results. *M. Holt.*

Godunov, S. K. Difference method of computation of shock waves. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 1(73), 176–177. (Russian)

This brief note emphasizes the usefulness of the methods of von Neumann and Richtmeyer [J. Appl. Phys. **21** (1950), 232–237; MR **12**, 289] and of Lax [Comm. Pure Appl. Math. **7** (1954), 159–193; MR **16**, 524] in solving, problems involving shock waves and other discontinuities. *M. Holt* (Providence, R.I.).

Žukov, A. I.; and Každan, Ya. M. On the motion of gas under the influence of a short impulse. Akust. Ž. **2** (1956), 352–357. (Russian)

The equations of one dimensional unsteady motion of a cold gas are integrated in Lagrangian form, under a hypothesis of self similarity. The boundary conditions correspond to an impulse applied at an open boundary of the gas, which is initially at rest with uniform density ρ_0 and zero pressure. Pressure, velocity and density vary according to the relations

$$p=A\rho_0 M^{-n}f(\eta),\quad u=\sqrt{A}M^{-n/2}w(\eta),\quad \rho=\rho_0 q(\eta),$$

where M is the mass of gas between the free boundary and shock wave, n is the self-similarity constant, and $\eta=m/M$, (m is the mass of gas at time $t=0$). From results of the integration curves of pressure and velocity against distance are plotted for different times. *M. Holt.*

Hida, Kinzo. Further note on the detached shock wave in front of a circular cylinder. Bull. Univ. Osaka Prefecture. Ser. A. **4** (1956), 37–43.

A Prandtl-Glauert linearization is used to study the flow immediately behind a detached shock wave in front of a circular cylinder and to obtain the separation of the detached shock from the cylinder. The procedure differs from a similar analysis by Laitone and Pardee [NACA Res. Memo. no. *A7B10* (1947)] in that the character of the irrotational flow behind the shock wave is approximately considered. The predictions for the shock detachment distances do not check with published experimental data, but they are an improvement over previous work of Laitone and Pardee. *P. Chiarulli* (Chicago, Ill.).

Rogers, M. H. Analytic solutions for the blast-wave problem with an atmosphere of varying density. Astrophys. J. **125** (1957), 478–493.

Consider a spherical shock wave of infinite strength at $R=(\eta_1 t)^{1/\lambda}$ that advances into an atmosphere with initial density $\rho_0(R)=\rho_e R^{-\alpha}$, where ρ_e, η_1, $\alpha<3$, and $\lambda=\frac{1}{2}(5-\alpha)$ are constants. Then the equations of spherically symmetrical flow have "similarity" solutions for velocity u, pressure p, and density ρ which satisfy the Rankine-Hugoniot conditions at the blast wave and are of the form $u(r,t)=Vf(x)$, $\rho(r,t)=\rho_0(R)h(x)$,

$$p(r,t)=\rho_0(R)V^2 g(x)/\gamma,$$

where $x=r/R$ and $V^2=(dR/dt)^2=4R^{\alpha-3}/\eta_1^2(5-\alpha)^2$. The total energy carried by the wave is constant. The functions f, g, h satisfy a first order system of ordinary differential equations which can be reduced to a single first order equation for f alone. The nature of the integral curves $f=f(x)$ depends on the relation of α to $\alpha_e=(7-\gamma)/(\gamma+1)$. For $3>\alpha>\alpha_e$ there may be density or pressure discontinuities on the sphere $x=x_1=f(x_1)$. Since there is no flow across $x=x_1$, cavitation will occur in such cases. Other types of similarity solutions, some including gravitation, are also briefly discussed. *J. Giese.*

Chester, W. Supersonic flow past a bluff body with a detached shock. I. Two-dimensional body. J. Fluid Mech. **1** (1956), 353–365.

This paper considers the flow behind the detached shock wave in front of a two-dimensional, symmetrical, round-nosed body in a uniform supersonic stream at zero incidence. With the use of the stream function as an independent variable, a closed form solution is obtained for the case $\gamma\to 1$, $M\to\infty$, $(\gamma-1)M^2$ finite, where γ is the adiabatic coefficient and M is the free stream Mach number. This solution, after suitable modification to make it uniformly valid in the neighborhood of the body, is used as the basis of a double series expansion in $\delta=(\gamma-1)/(\gamma+1)$ and M^{-2}. The expansion is carried out to within terms of order $(\delta+M^{-2})^2$. Calculations have been carried out (radius of curvature at body nose equal to one) to obtain the distance between the shock apex and body nose, the radius of curvature at the shock apex, the ordinate of the sonic point on the body and on the shock, and the pressure distribution on the body. Among other results, it is found that the shock stand-off distance is essentially determined by the parameter $\delta+M^{-2}$ and does not depend upon the explicit values of M and γ. The check with experimental data is not good, except for the pressure at the body near the nose. *P. Chiarulli.*

Chester, W. Supersonic flow past a bluff body with a detached shock. II. Axisymmetrical body. J. Fluid Mech. **1** (1956), 490–496.

The techniques described in Part I [see the paper reviewed above] are used to consider the axisymmetrical case. Convergence of the resulting expressions is poorer than that in the two-dimensional case. *P. Chiarulli.*

Kuerti, G.; and Faymon, K. One-dimensional shock-motion of a gas with $\nu=3$, enclosed between fixed walls. J. Math. Phys. **35** (1957), 384–393.

This concerns an example of gas motion in a tube closed at both ends in which conditions are isentropic, except across shock waves, even though the velocity is non-uniform. The variation of the physical variables along the tube at successive times is calculated and the shock path is followed from its formation through three reflections. *M. Holt* (Providence, R.I.).

Černyĭ, G. G. The problem of point explosion. Dokl. Akad. Nauk SSSR (N.S.) **112** (1957), 213–216. (Russian)

The author seeks solutions of the Lagrangian equations of motion for plane, cylindrical, or spherically symmetrical flow in the form $R=R_0+\varepsilon R_1+\cdots$, $p=p_0+\varepsilon p_1+\cdots$, $\rho=\rho_0+\varepsilon\rho_1+\cdots$, where $p(r,t)$ is pressure, $\rho(r,t)$ density, $R(r,t)$ distance from the center for the particle originally at distance r, and $\varepsilon=(\gamma-1)/(\gamma+1)$. The equations of motion and the boundary conditions at the shock wave

$R=R_0(t)$ lead to explicit functional forms for p_0, ρ_0, p_1, ρ_1, and R_1 in terms of R_0, dR_0/dt, and d^2R_0/dt^2. Then the equation of conservation of energy, including that of the explosion yields an integro-differential equation for $\varphi(x)=(dR_0/dt)^2$ where $x=R_0^\nu/\nu$, and $\nu=1, 2, 3$ is the number of dimensions of the type of flow considered. The author determines two initial terms of a solution $\varphi=(\tilde{\alpha}(\gamma,\nu)x)^{-1}+A(\gamma,\nu)+\cdots$ and shows that these lead for $\nu=3$ to better agreement with the pressure ratio at the shock calculated by H. Goldstine and J. von Neumann [Comm. Pure Appl. Math. **8** (1955), 327–353; MR **17**, 314] than approximations by progressive waves or by linearized theory. *J. Giese* (Aberdeen, Md.).

Frederiksen, Eyvind. Resonance-behavior of non-linear one-dimensional gas vibrations analyzed by the Ritz-Galerkin method. Ing.-Arch. **25** (1957), 100–112.

An analysis is made of the response of a volume of gas to harmonic oscillations of a piston at one end. Experimental results due to E. Lettau [Deutsche Kraftfahrtforschung **39** (1939)], showing that the resulting vibrations are non-linear and may lead to shock waves, are largely confirmed. *M. Holt* (Providence, R.I.).

Stewartson, K. The dispersion of a current on the surface of a highly conducting fluid. Proc. Cambridge Philos. Soc. **53** (1957), 774–775.

In the region $x<0$ there is empty space with zero electrical conductivity and in the region $x>0$ there is an inviscid incompressible fluid of conductivity σ. Suppose that the static magnetic field in $x<0$ is $\mathbf{H}_0=(H_0, 0, 0)$ and in $x>0$, $\mathbf{H}_1=(H_1, 0, 0)$. By considering oscillations about this state the author shows that in the limit $\sigma\to\infty$, the discontinuity in the magnetic field would be dispersed instantaneously, the discontinuity moving as a shock with the Alfvén velocity. *S. Chandrasekhar.*

Nardini, Renato. Sui fronti d'onda nella magneto-idrodinamica. Riv. Mat. Univ. Parma **7** (1956), 3–32.

Cet article constitue une étude détaillée de la propagation des ondes sonores dans les fluides doués de conductivité électrique. Les équations du problème sont d'une part les équations de Maxwell dans lesquelles figure la composante $\mathbf{v}\times\mu\mathbf{H}$ du champ électrique dûe au mouvement des particles fluides ($\mathbf{v}$ vitesse du fluide, $\mathbf{H}$ champ magnétique, μ perméabilité magnétique) d'autre part l'équation de la dynamique dans laquelle figurent les forces électromagnétiques $\sigma(\mathbf{E}+\mathbf{v}\times\mu\mathbf{H})\times\mathbf{H}$ ($\mathbf{E}$ champ électrique, σ conductivité électrique) et la conservation de la masse. La vitesse de propagation des ondes sonores est déterminée par l'étude des variétés caractéristiques du système, lorsque la viscosité est nulle et la conductivité électrique infinie. Pour un fluide incompressible on généralise ainsi un résultat dû à Alfvén [Cosmical electrodynamics, Oxford, 1950; MR **12**, 765]. L'auteur traite ensuite le cas d'un fluide compressible et obtient en chaque point dans chaque direction la valeur des deux principales vitesses de propagation; le système, totalement hyperbolique, comportant huit équations du premier ordre il existe deux autres vitesses de propagation, l'une est nulle, l'autre indépendante de la compressibilité; elles apparaissent dans les articles fondamentaux dûs à de Hoffmann et Teller [Phys. Rev. (2) **80** (1950), 692–703] et à Friedrichs (Los Alamos Laboratory Scientific Report, 1954) dont l'auteur ne semble pas avoir eu connaissance. *H. Cabannes* (Marseille).

Nardini, Renato. Su qualche effetto del secondo ordine nella magneto-idrodinamica. II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **20** (1956), 591–596.

Pai, Shih-I. Energy equation of magneto-gas dynamics. Phys. Rev. (2) **105** (1957), 1424–1426.

L'équation de conservation de l'énergie est écrite dans le cas général d'un fluide doué de viscosité, de conductivité thermique et de conductivité électrique. Diverses formes simplifiées sont discutées, les équations fondamentales de la dynamique des gaz doués de condutcivité electrique sont rappelées. *H. Cabannes* (Marseille).

Crausse, Étienne; et Poirier, Yves. Sur l'écoulement laminaire d'un liquide électriquement conducteur soumis à l'action d'un champ magnétique transversal. C. R. Acad. Sci. Paris **244** (1957), 2694–2695.

Les auteurs envisagent l'écoulement dans une conduite rectiligne d'un liquide doué de conductivité électrique et soumis à un champ magnétique transversal. Par des considérations dimensionnelles ils expriment le gradient longitudinal de pression. Dans le cas particulier d'un écoulement entre plaques parallèles éloignées, étudié par J. Hartmann [Danske Vid. Selsk. Math.-Fys.Medd. **15** (1937), no. 6] les puissances volumiques dissipées par diffusion (viscosité) et par effet Joule sont calculées.

H. Cabannes (Marseille).

Richter, Egon. Erweiterung der Theorie magnetohydrodynamischer Wellen und Anwendung auf inhomogene Schichten. Z. Naturf. **11a** (1956), 901–912.

Philip, J. R. Transient fluid motions in saturated porous media. Austral. J. Phys. **10** (1957), 43–53.

In this paper the transition from rest to steady motion on the sudden application of a potential gradient to the fluid contained in a saturated porous medium is investigated. Exact solutions of the Navier-Stokes equations (with the quadratic terms neglected, the Reynolds number can be expected to be small) are compared with an approximate analysis using macroscopic velocities based on the assumptions that the time-dependent velocity is a function of time times the steady state velocity. It is found that the error in using Darcy's law will usually be unimportant for the case of a suddenly applied potential gradient.

The approximate solution analysis is extended to include a general time variation of the applied potential gradient. The author concludes that when the applied potential gradient is periodic there may be significant deviations from Darcy's law. *R. C. DiPrima.*

See also: Numerical Methods: Nonweiler; Batschelet and Grün. **Probability:** Longuet-Higgins. **Astronomy:** Keller.

Optics, Electromagnetic Theory, Circuits

★ **Buchdahl, H. A. Optical aberration coefficients.** Oxford University Press, London, 1954. xx+336 pp. $8.00.

In recent years the great demand for precision optical

instruments of all types has led to an increased activity of studying in a systematic way the theoretical and practical aspects of aberrations of optical systems. In the practical design of an optical instrument of precision one must make a careful analysis of the geometrical and chromatic aberration of the system. Therefore a general knowledge of the theory of aberrations is expected of the modern designer of optical instruments. The three methods commonly used in studying aberrations are: (a) The method of characteristic functions due to Hamilton, (b) the trigonometric method of ray tracing and (c) the purely algebraic method. In practice it might be convenient to use a combination of these methods. The first method is the most general and fruitful one from the theoretical point of view, but hardly the most useful to a practical designer with limited mathematical equipment [see, e.g., M. Herzberger, Strahlenoptik, Springer, Berlin, 1931]. Therefore to the practical designer the trigonometric and the algebraic methods seem to be of greater usefulness in designing optical instruments. With an eye to the practical designer the present book contains the most complete and systematic treatment of geometrical aberrations (g.a.) of symmetric optical systems which has so far appeared in book form, based on the algebraic method. The aim of the author has been to present in a moderate size volume the essentials and the procedures for calculating the aberrations. By introducing different kinds of variables and coordinates (canonical, paracanonical, etc.) the expressions for the aberration coefficients (a.c.) yield to a more systematic and uniform treatment than otherwise. Even then, the expressions for the a.c. of order higher than the third are rather lengthy and in a way formidable-looking to a beginner. Unfortunately, this situation is inherent in the nature of the subject and the author has tried to present it in as clear a manner as possible.

In order to give some idea of the contents of the book a partial description of the material by chapters will suffice. The book is divided into three parts; I. Monochromatic aberration coefficients; II. Chromatic aberration coefficients; and III. Variational methods for adjusting monochromatic and chromatic a.c. resulting from changes in the components of the optical systems. In Part I the author begins with the definition of geometrical aberration coefficients (g.a.c.) from the algebraic standpoint, their representation in terms of canonical and paracanonical variables, invariants and quasi-invariants, including iteration processes for calculating higher order a.c., intrisic a.c. for spherical surfaces, identities between a.c., Seidel aberrations of third and fifth order, dependence of a.c. on the position of pupil planes and object and image planes. A full account of seventh and higher order a.c., with the emphasis laid in computing methods, and a detailed analysis of an actual optical system (Cooke's triplet) with extensive tables and curves for various order aberration coefficients, are given. This is followed by a detailed analysis of aspherical surfaces.

Part II deals with chromatic aberrations. The displacement of the image point from the ideal one is expressed in the form of a power series in terms of the usual three optical invariants and the so-called colour variable (invariant) representing wave length differences. The coefficients of this quadruple series are the general a.c. A distinction is made between the chromatic aberration coefficients (c.a.c.) and the chromatic variation of monochromatic aberration coefficients, the latter being a special form of the former. The third and fifth order chromatic spherical aberration and coma are derived. Chapter XI contains an analysis of the identities between c.a.c. for paraxial and non paraxial rays. In Chapters XII and XIII special cases dealing with the c.a.c. and their dependence on the position of pupil and object and image planes are worked out in great detail, and numerical values of these are given in a number of tables illustrating the method of computing them. Part III begins with a detailed treatment of the method of variation of parameters for studying the overall problem of the dependence of the resultant aberrations on the structure of the optical system, i.e., the dependence of a.c. on the curvatures and separations of the surfaces (lenses, etc.) on the refractive indices of the components, on the effect of dispersion and temperature, etc. The results are applied to third and fifth order aberrations and seventh order spherical aberration. Due emphasis is placed to spherical aberration, which places a prominent role in designing quality instruments, and on extra-axial curvature. In chapter XIX the author has worked out the effect of variation of monochromatic and chromatic parameters on the chromatic aberrations coefficients, in particular on the third and fifth order. Among other topics one also finds a discussion and analysis of the dependence of third and fifth order paraxial a.c. on the the same order dispersion coefficients of a given component, and the effect of temperature and pressure. The last chapter contains extensive tables illustrating the general method. In the appendices the author explains some points which have been only mentioned in the text. Of particular interest, one finds in appendix *D* a procedure of how to find the angular characteristic from the knowledge of the displacement of the image point from the ideal one, and in appendix *F* a brief treatment of non-homogeneous media is described.

It is the opinion of the reviewer that the long development of fifth and seventh order aberration could have been left out or treated briefly, without impairing the usefulness of the book. In its place the author could have included an account of the trigonometric method with a discussion and a comparison of the merits of the two methods. Furthermore one cannot find a trace of the recent important work of the Dutch and French schools, especially that of Zernike and his pupils, who have developed the theory of aberrations from the Hamiltonian method. However, these shortcomings are compensated by the thoroughness and painstaking care which the author has applied in presenting to the public the algebraic theory of aberrations of symmetrical optical systems. For the optical practitioner this book supplies a wealth of material in helping him to design quality instruments, and on the other hand it should be of interest to the optical theoretician and applied mathematician.

N. Chako (Flushing, N.Y.).

Buchdahl, H. A. Optical aberration coefficients. I. The coefficient of tertiary spherical aberration. J. Opt. Soc. Amer. **46** (1956), 941–943.

In the book reviewed above the author has explicitly determined the coefficient of tertiary (i.e. 7th order) spherical aberration and developed a calculational scheme involving 121 entries per surface. The present paper introduces simplification so that only 19 entries per surface are now required. The proofs are phrased in the powerful notation of the author's book, but a long appendix abstracts the necessary definition so that no reference is necessary. *G. L. Walker.*

Cap, F. Le principe de Fermat dans les milieux absorbants non-homogènes. Ann. Inst. H. Poincaré **15** (1956), 123–131.

In Maxwell's equations for a medium with conductivity κ, the author substitutes $\mathbf{E}=\mathbf{E}_0 \exp ik(ct-S-iT)$, where S and T are real functions of position, and obtains, for k large, the approximate equations

$$(*) \qquad (\nabla S)^2-(\nabla T)^2=n^2-\kappa^2, \quad \nabla S\cdot\nabla T=-n\kappa,$$

where n is the real part of the complex index of refraction. The rays, defined as the orthogonal trajectories of $S=$ const., obviously satisfy $\int|\nabla S|ds=\min$, but, since T cannot in general be eliminated from (*), this Fermat principle does not yield differential equations for the rays. However, for a stratified medium we have $T=T(x)$ and the elimination can be effected; the rays in the (x, y)-plane satisfy $\int m\,ds=\min$, where

$$m^2=\tfrac{1}{2}(n^2-\kappa^2)+[\tfrac{1}{4}(n^2-\kappa^2)^2+n^2\kappa^2(1+y'^2)]^{\frac{1}{2}}.$$

J. L. Synge (Dublin).

Summarized proceedings of a conference on contemporary optics, Sydney, 1956. J. Sci. Instrum. **34** (1957), 129–135.

This paper reviews a conference held in Sydney in September 1956 and reports mainly on papers of Australian workers dealing with the following subjects. 1) Geometrical optics: Methods for completing the design of an optical system from a rough design, design of triplets, the computation of third and higher aberration coefficients, reflecting microscopes. 2) Radiative transfer: Methods of solution of multi-dimensional radiative transfer problems. 3) Spectroscopy: Developments of atomic absorption spectrophotometers for chemical analysis, the application of diffuse reflexion spectrophotometry to chemical analysis, techniques for the generation of accurate cylinders required in the construction of diffraction gratings by the Merton method. 4) Physical optics: Theory and measurement of contrast transfer functions, imagery with partially coherent illumination, transfer functions in radio astronomy, techniques for the treatment of diffraction in small aperture multi-component systems, communication theory in optics. 5) Interferometry: Influence of the position and size of the effective source on the fringes produced in interferometer systems using collimated light, microsecond sources for interferometry, the determination of the optical constants of ionic solids. *E. Wolf.*

Lenz, F.; und Scheffels, W. Über die Veränderung des Streuvermögens elektronenmikroskopischer Objekte unter Elektronenbeschuss. Z. Naturf. **11a** (1956), 656–662.

In order to obtain electron-optically, in the image plane, a good contrast between various parts of an object (thin foils), one places the object in a caustic region and in front of the focal plane of an electron microscope so that the object is in a region where the fields vary rapidly (strongly) and the current density changes very rapidly with the position of the object plane. In this paper the authors have investigated quantum mechanically the diffraction and scattering properties of an electron beam by thin foils. By expressing the electron rays in a plane wave form, the scattering function due to elastic scattering by atoms is given by Born's formula (first approximation). There is an additional phase factor not included in the first Born approximation. The knowledge of the latter is important when one considers diffraction effects. For N similar scattering centers (atoms) an expression for the current density is obtained if the atoms are not in motion (solids and low temperatures). For amorphous substances one has random distribution of atoms and thus real substances lie between these two extremes, the amorphous state and the crystal state. By considering both types of arrangements, expressions for the scattering current density are obtained. The term containing the square of the number of scatterers is due to coherent scattering and the second term depending linearly on the number of scatterers arises from the incoherence effect, which results from the amorphous state. This of course is not true for solid films which would give rise to interference effects (interatomic scattering). The author consider two models, spherical and quadrilateral scatterers. By taking account of inelastic scattering the authors have derived an expression for the current density which seems to be primarily incoherent scattering. Finally the dependence of coherent and incoherent scattering on the state of excitation of the object atoms has been analyzed provided the scattering angle is small, and the results are applied to carbon atoms. *N. Chako* (Flushing, N.Y.).

Picht, J. Zur beugungstheoretischen Behandlung elektronenoptischer Abbildungsfehler. Optik 13 (1956), 494–501.

In this paper the author has formulated the problem of electron optical aberrations from the point of view of difraction theory. In an important paper [Ann. Physik (4) **77** (1925), 685–785; Optische Abbildung, Vieweg, Braunschweig, 1931, chap. 8] the author derived, for the ordinary optical systems with aberrations, an integral expression for the amplitude function at a point P in image space. In analogy to the optical case the author proposes that this integral expression, known at present as the Debye-Picht-Luneberg formula, namely

$$\text{(i)} \quad \psi_P=\frac{ik}{2\pi}\iint A(u, v)\exp[-ik(P-W(u, v)\cdot n(u, v))]d\Omega$$

can be carried over to electron optical systems provided the quantities in the integrand are properly interpreted. The expressions $A(u, v)$, P, $W(u, v)$ and $n(u, v)$ are respectively, the amplitude of the wave on the surface of integration (s.i.) or wave front; the vectorial distance from some point Q (object point) to the point P where the amplitude is to be calculated; the normal vector associated with the wave surface (w.s) (Hamilton's mixed characteristic) and finally n is the normalized directional vector in the direction of the normal of W. The expression $P\text{-}W\cdot n$ is the optical disance from the s.i. to the image point (Aufpunkt) P. The angular spread of the ray bundle formed at P by an element of the s.i. $d\sigma$ is denoted by $d\Omega$. To simplify matters one chooses a wave surface (front) W, called the reference w.s. in such a way that the caustic of the ray bundle lies on one side of it, i.e. the rays of the bundle are uniquely determined, In order that (i) can be applied to electron optical aberration systems the following conditions must be fulfilled: (a) The image plane containing the point P must lie in a region where the fields give rise to a constant refractive index, or are such that in an extended region the refractive index remains sufficiently constant and the reference w.s. lies in this region; (b) if this is not satisfied, then one assumes the electron optical medium with the plane (image) adjoins a homogeneous isotropic medium having a constant refractive index. The difficulty of applying (i) directly to electron optics comes from the fact that on one side of

the free aperture (screen) of the bundle, the rays cannot be associated with plane waves propagating in the directions of the rays which is the case for homogeneous isotropic media. In electron optics the wave surfaces are curved and the propagation vector does not coincide with the ray vector. The author considers a plane wave (in non-homogeneous anisotropic medium) of the form

(ii) $$\psi=a(r)\exp\Big[-2\pi i\nu\Big(t-\frac{(r\cdot n)}{u_{r,n}}\Big)\Big],$$

$u_{r,n}$ is the phase velocity at a point defined by the vector r (distance) in the direction n. By using the relations, $E=h\nu$, $E_0=m_0c^2$ one writes (ii) as

$$\psi=a(r)\exp[-2\pi i/h(Et-E_0n_{r,n}(r\cdot n)/c)].$$

The surfaces of constant phase are given by $n_{r,n}(r\cdot n)=$ constant and the refractive index $n_{r,n}$ is determined from the fields at the point, n being a constant unit vector. The various directional coordinates n are determined from the w.s. $W_{P,n}$ which pass through P (Aufpunkt or image point) and r can be identified with the point P. One chooses the reference w.s. W_k associated with the ray bundle from the object side point 0. This w.s. can be determined from the point characteristic, namely $W=S-Et$ or from the relation

(iii) $$\int_0^{W_*}\frac{n\cdot dr}{u}=\frac{E_0}{c}\int_0^{W_*}n_{r,n}ndr=\text{constant}.$$

From these considerations formula (i) can be carried over to electron optical systems (non-homogeneous anisotropic media). The result obtained by the author for the amplitude at the image point P is

(iiii) $$\psi_P=\frac{iE_0}{hc}\iint n_{p,N}A_{Ne}-\frac{2\pi iE_0}{hc}\int_{W_*}^{W_{P,n}}n_{ds}dsd\Omega$$

where the integration has to be carried over all possible directions N belonging to the amplitude A_N.

N. Chako (Flushing, N.Y.).

Cowley, J. M.; and Moodie, A. F. The scattering of electrons by atoms and crystals. I. A new theoretical approach. Acta Cryst. **10** (1957), 609–619.

Dynamic theory of scattering of electrons by crystals is at present tractable only when two or three strong beams exist in the crystal, but not for a large number of beams. A new theory is described which makes use of the Huygen's Principle to calculate the amplitude distribution on one plane in terms of a distribution on preceding planes. A small-angle approximation is made, which replaces spherical wave-front by a paraboloid. The three-dimensional potential field is approximated by a set of two-dimensional potential distributions. The wave function is obtained on an arbitrary plane of observation. Particular cases are the electron microscope image and the diffraction pattern of a crystal. The theory is suited for study of diffraction by thin crystals and crystals containing imperfections. *V. Vand* (University Park, Pa.).

Wiskott, D. Zur Theorie des Auflicht-Elektronenmikroskops. I. Optik **13** (1956), 463–478.

In order to observe electron-optically the surface structure of a conducting surface (object), the experimental set up must be arranged in such a way that the surface object in combination with one or more aperture screens should behave like an electron optical mirror. In this article the author has investigated the geometrical electron optical properties of a system composed of a conducting surface (object) and single aperture screen. The potential of the surface is maintained as close as possible to the cathode potential and such that to a first approximation the field between the conducting surface and the aperture is constant. The 3-dimensional potential of the system is separated into two parts, (a) a potential which gives the homogeneous field (asymptotic value) and a perturbing field which vanishes at the cathode. For short range (short lens) and a weak perturbing field the approximate equations of the paths in the x-y plane are derived on the assumption that the axial coordinate is a parabolic function of the time. For a special case (grooved surface with a potential distribution of the form $\varphi=z-z_0+z(z^2+x^2)^{-1}$ where z_0 denotes the location of the surface (object) the equations of the paths are given in explicit form. The action of the groove for short distances from the groove is similar to that of a short lens. The electron paths, the caustic surface and the current distribution in the x-z plane are plotted and photographs of the structure of a groove of width 10μ and depth 2μ are obtained. It is found that the contrast is good when the voltage of the object is close to zero. In the second part the author treats the case when the disturbing potential is appreciable. From physical considerations the disturbing potential must be a regular function for $z>0$ and must satisfy the condition, $\varphi(x, y, z)=0$ as $z\to\infty$, for all x and y. Under those conditions the two-dimensional $\varphi(x, z)$ is expressed as the real part of a Laplace transform, whereas the three-dimensional potential is expressed by a Fourier series involving integral expressions of Bessel functions. By assuming a power series development of the kernel (both cases) the perturbed potential fields are evaluated. Physically these expressions correspond to multipole expansion in two and three dimensions. The current distribution is explicitly evaluated for the two dimensional case. *N. Chako* (New York, N.Y.).

Wiskott, D. Zur Theorie des Auflicht-Elektronenmikroskops. II. Optik **13** (1956), 481–493.

In this paper the author has studied the wave analogue of the problem treated in the paper reviewed above based on Schrödinger's equation. By assuming a double periodic expansion of the perturbed function $\psi(x, y, z)$ in x and y, and an exponential decay in z, Schrödinger's equation reduces to a two parameter family of coupled ordinary equations of the Airy type in the z coordinate. The integration of the system can be affected by applying the method of successive approximation. The zero approximation, $U_\infty(z)$, is given by Airy's functions with arguments depending on the periodicity of the perturbed potential. The reflected wave and the current density are obtained by taking the asymptotic form of the Airy functions. As a special case, the author considers the leading term of the double series with equal periodicity in x and y and calculates the ratio of the maximum current density to the minimum current density of the outgoing wave, and an expression for the resolution of the periodic structure of the object is given in simple form when the first approximation to the wave function is considered. For an accelerating potential of 100 KV/cm, the resolution is about 80 Å units. The next section deals with the geometrical solution of this problem which is the limiting case of the wave mechanical treatment. A comparison of the results of the two methods is discussed and illustrated. *N. Chako* (New York, N.Y.).

Tretner, W. Zur unteren Grenze des Öffnungsfehlers magnetischer Elektronenlinsen. Optik **13** (1956), 516–519.

In this paper the author gives a lower limit for the aperture error of a magnetic lens. Such a limit exists, if the axial magnetic field $h(z)$ vanishes for some finite value of z. With the aid of formulas derived in previous papers [Optik **11** (1954), 312–326; **12** (1955), 293], the author shows that the aperture error C_3 is given by the relation, $C_3=0.6[h_0'(z)]^{-\frac{1}{4}}$, where $h(z)=zh_0'$. For optimum fields of the form, $h=(h_0')^{\frac{1}{2}}f(z, h_0')$, the aperture error is of the form $C_3=0.239|H/H'|$, when the image lies either at a finite or infinite distance, with $H=h[8mU/e]^{\frac{1}{2}}$ and where U is the accelerating potential. If the object is located at one of the zeros of h or in free space, then for any magnification the error, $C_3\geqq 0.6[8mU/e]^{\frac{1}{4}}(H_{\max}')^{\frac{1}{2}}$. *N. Chako.*

Grümm, H.; und Spurny, H. Beispiele streng berechenbarer Ionen- und Elektronenbahnen im ebenen elektrostatischen Feld. Acta Phys. Austriaca **11** (1957), 11–22.

By making use of the Staeckel criterion for the separation of the Hamilton-Jacobi equation (method of separation of variables), the authors have treated the following cases: (a) a potential field which depends linearly on one coordinate; (b) a potential field of hyperbolic form in $x-y$; (c) a general potential field in two dimensions of the form such that in the first approximation it reduces to a hyperbolic field. The electronoptical properties of these fields have been worked out in detail. Finally by considering a potential field of the type

$$\varphi(u, v)=a(u^2-v^2)+(bu+cv)/u^2+v^2)+d,$$

where u and v are parabolic-cylindrical coordinates defined by $u=\sqrt{(x^2+y^2+x)}$, $v=\sqrt{(x^2+y^2-x)}$, for which the Hamilton-Jacobi equation is separable, the paths in the x-y plane are found to be cardioids, if $a=d=0$.

N. Chako (New York, N.Y.).

Morris, Chester R. Electron trajectories in a combined electric and magnetic field. Math. Mag. **30** (1957), 251–267.

In this article the author has studied the motion of an electron under the influence of a combined potential field of the form $\varphi=\frac{1}{2}a(x^2+y^2-2z^2)$ and a uniform magnetic field in the z direction. The differential equations reduce to a coupled system in x and y and a harmonic type equation for the z direction. The general solution of these systems of equations are well known. The author has given various solutions of these equations for a number of cases depending on the initial values of the coordinates and the velocities of the electron and for different magnitudes of the magnetic field. *N. Chako.*

Kummerer, Karl. Zur Theorie der Elektronenbremsstrahlung im Gebiete thermischer Elektronengeschwindigkeiten. Z. Physik **147** (1957), 373–394.

Sommerfeld's expression for the radiation loss of a current of electron in the Coulomb field of a point charge is essentially given by the differential quotient of the square of a (Gauss) hypergeometric function. This process is performed after a rearrangement of Sommerfeld's formula. The second approximation of the result is considered. An estimate for the remainder is established and a comparison with the result of other authors is given.

F. Oberhettinger (Washington, D.C.).

Bomze, Josef. Über die Möglichkeit der Existenz von konservativen elektrischen Ladungsbewegungen mit nicht-stationären Feldern im Rahmen der klassischen Elektrodynamik. Österreich. Akad. Math.-Nat. Kl. S.-B. II. **165** (1956), 313–325.

The two d'Alembert solutions of the inhomogeneous wave equation for the electromagnetic potentials are investigated. It is shown, that the flow of energy of a system of moving electric takes place either in the exterior or in the interior direction according to which one of the two d'Alembert solutions is chosen. The simultaneous use of both solutions admits the possibility of the existence of moving charge with non-stationary fields without any radiation of energy.

F. Oberhettinger (Washington, D.C.).

De Socio, Marialuisa. Sulla velocità dell'energia in un gas ionizzato soggetto ad un campo magnetico. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **90** (1955–56), 563–566.

Van Mieghem has shown in "Propagation des ondes électromagnétiques en milieu homogène" [Gauthier-Villars, Paris, 1944] that the energy velocity in a plane wave propagating in an ionized gas not being subject to a magnetic field, coincides with the group velocity. In another of his works he generalized this result to the case of an ionized gas subject to a magnetic field, provided that the direction of motion of a plane wave is parallel or perpendicular to the direction of the magnetic field. In the present paper the writer considers the second case with the direction of motion of a plane wave being at any arbitrary angle to the direction of the magnetic field and demonstrates the validity of the following theorem: the component of the mean velocity of the energy in the direction of the propagation of the wave coincides with the group velocity. *M. Z. v. Krzywoblocki.*

Wilcox, Calvin H. Debye potentials. J. Math. Mech. **6** (1957), 167–201.

If u and v are solutions of the partial differential equation (1) $\nabla^2u+k^2u=0$ in a domain D and if $\mathbf{r}$ is a position vector relative to an origin outside D, then the vector fields (2) $\mathbf{A}=\nabla\times(\nabla\times u\mathbf{r})+k\nabla\times v\mathbf{r}$, $\mathbf{B}=\nabla\times(\nabla\times v\mathbf{r})+k\nabla\times u\mathbf{r}$ satisfy Maxwell's equations (3) $\nabla\times\mathbf{A}=k\mathbf{B}$, $\nabla\times\mathbf{B}=k\mathbf{A}$ for a time-harmonic electromagnetic field in D: u and v are called the Debye potentials of that electromagnetic field. The author proves that every electromagnetic field defined in a region between two concentric spheres can be represented in the form (2). His proof is based on a decomposition theorem for a field of tangent vectors on the surface of a sphere which is a special case of Hodge's decomposition for fields of p-vectors on a Riemannian manifold in an exact, coexact and harmonic field.

A vector field such as $\mathbf{A}$ above is called a vector wave function. If it satisfies an appropriate radiation condition (expressing that it corresponds to "outgoing" waves) it is called a vector radiation function. The author expands u and v in (2) in series of spherical surface harmonics and thereby obtains an expansion of vector radiation functions defined in a domain bounded by concentric spheres which may be interpreted as the expansion of an electromagnetic radiation field in a series of electromagnetic multipoles. He next shows that every vector wave function with an isolated singularity in D may be decomposed in a wave function which is regular throughout D and a radiation function. By applying the expansion theorem to the radiation function, he is able to analyze the nature of

isolated singularities of an electromagnetic field. He shows, inter alia, that there is no isolated singularity of order less than two, and that only isolated singularity of order less than three of **A** is that of the field of a magnetic dipole. *A. Erdélyi* (Pasadena, Calif.).

Gould, R. N.; and Cunliffe, A. On the coupling between two cavities. Phil. Mag. (8) **1** (1956), 1126–1129.

The authors describe a general method of calculating the oscillatory properties of iris-coupled cavity resonators. Their method resembles the integral equation technique used by Schwinger [Mass. Inst. Tech., Radiation Lab. Rep. **43–34** (1943)]. *C. H. Papas* (Pasadena, Calif.).

Cazenave, René. Un paradoxe en électrostatique. Cahiers de Phys. no. **78** (1957), 89–92.

This paper considers the interaction between a point charge and an insulated conducting sphere with a charge of the same sign as the point charge. It is found that when the two are sufficiently close, the sphere will attract the point charge. The paradox appears to be connected with the purely fictitious character of the influencing point charge. *J. E. Rosenthal* (Passaic, N.J.).

De Socio, Marialuisa. Sulla propagazione delle onde elettromagnetiche in un gas ionizzato soggetto a un campo magnetico. Atti Sem. Mat. Fis. Univ. Modena **7** (1953–54), 68–77 (1956).

The known equations for propagation of electromagnetic waves through the ionosphere, subject to the earth's magnetic field, are rederived on the basis of the following picture. The charged particles have no mutual collisions, but are slowed down by friction, presumably caused by collisions with neutral particles. The equations are linearized by neglecting the influence of the oscillating magnetic field on the particles. The resulting set of linear differential equations with constant coefficients gives rise to two different modes of propagation. (Recently J. H. Piddington [Phil. Mag. (7) **46** (1955), 1037–1050; MR **18**, 170] found four kinds of waves; he considered the mutual collisions preponderant, and neglected collisions with the background.) *N. G. Van Kampen* (New York, N.Y.)

Bak, M. A.; Petržak, K. A.; and Romanov, Yu. F. Radiation from a spherical source in the presence of self-absorption. Ž. Tehn. Fiz. **26** (1956), 379–384. (Russian)

The author calculates the radiation due to a uniform source density distribution within a sphere, when the medium within that sphere is lossy. All interference, diffraction, and refraction effects are neglected as the calculations are apparently intended for interpretation of measurements of radioactivity. *J. Shmoys.*

★ **van de Hulst, H. C. Light scattering by small particles.** John Wiley and Sons. Inc., New York; Chapman and Hall, Ltd., London, 1957. xiii+470 pp.

The techniques of the calculation of the scattering of waves from a variety of objects is a subject of considerable analytical difficulty but with a wide variety of important practical applications. The literature on the subject has been so widely dispersed that, when a new application turns up, it has often been easier to work out the theory anew than to find the previous calculations.

In the present volume is collected, for the first time, a discussion of the subject as a whole. The first part of the book is devoted to the basic theory, the conservation theorems, the fundamental formulas for multiple scattering and a discussion of the effects of polarization. The second part is devoted chiefly to the methods of calculation of scattering from spheres of various sizes and of various indices of refraction. On page 107, for example, is a clear statement of the relationship between Babinet's principle and the curious fact that the total cross section for scattering from a sphere goes to twice the intercepting area of the sphere, in the limit of very short wave-lengths. The third and last section is a discussion of some of the applications of the theory, to chemistry, to meteorology and to astrophysics.

Curves and tables are given of the more useful functions involved. The bibliographies, at the ends of the chapters, are fairly complete and should be helpful for those wishing further details. *P. M. Morse* (Cambridge, Mass.).

Hartenstein, Brigitte; und Friedl, W. Zur Intensitätsverteilung eines Plangitters mit dreieckigem Furchenprofil. Optik **14** (1957), 119–129.

The intensity distribution arising from the diffraction on a plane grating with a triangular groove profile is investigated and an earlier formula due to Hatcher and Rohrbaugh [see, e.g., J. Opt. Soc. Amer. **46** (1956), 104–110] is corrected. For the special case when the profile is symmetrical, the intensity curves relating to the first six orders, are given. *E. Wolf* (New York, N.Y.).

Biot, M. A. A new approach to the non-linear problems of FM circuits. Quart. Appl. Math. **15** (1957), 1–10.

Using the concept of instantaneous frequency and a Fourier representation of the input signal, the output of an FM receiver is expressed in terms of quotients of certain double integrals taken over the neighborhoods of certain points. In applications Fourier coefficients are evaluated by residue theory for analytic functions. A number of examples are worked out. *N. Levinson.*

Grinberg, G. A.; Lebedev, N. N.; Skalskaia, I. P.; and Uflyand, Ya. S. The electromagnetic field of a line source located in the interior of an ideally conducting parabolic screen. Translated by A. Shenitzer. Div. Electromag. Res., Inst. Math. Sci., New York Univ., Res. Rep. No. **EM-103** (1957), ii+27 pp.

Translated from Ž. Eksper. Teoret. Fiz. **30** (1956), 528–543.

★ **Miller, M. A.; and Talanov, V. I. Surface electromagnetic waves guided by boundaries of slight curvature.** Translated by Morris D. Friedman, 572 California St., Newtonville 60, Mass., 1957. 15 pp.

Translated from Ž. Tehn. Fiz. **26** (1956), 2755–2765.

Arus, Lorenzo. Sulle oscillazioni forzate nei sistemi non lineari a n gradi di libertà. Atti Sem. Mat. Fis. Univ. Modena **7** (1953–54), 182–188 (1956).

The systems considered are those appropriate to n electrical circuits coupled together by mutual inductances (but in no other way) and having non-linear resistances, the whole being driven by various sinusoidal electromotive forces with period T. Each circuit also contains a linear condenser and a selfinductance. On the assumption that the mutual inductances are negative, the author obtains upper bounds for the root mean square periodic currents which are independent of the resistances.

D. C. Lewis, Jr. (Baltimore, Md.).

★ **Fialkow, A. D. Networks without mutual reactance.** Proceedings of the Symposium on Modern Network Synthesis, New York, 1955, pp. 79–97. Polytechnic Institute of Brooklyn, Brooklyn, N. Y., 1956.

The author gives a resumé of the results he has obtained together with I. Gerst on the analysis and synthesis of RC and RLC networks which do not contain transformers [see MR **14**, 116; **15**, 377; **16**, 1182]. Of special concern is the transfer function for networks of the following classes: three-terminal, *L*, four-terminal, and lattice. *R. J. Duffin* (Pittsburgh, Pa.).

★ **Effertz, F. H. On the synthesis of networks containing two kinds of elements.** Proceedings of the Symposium on Modern Network Synthesis, New York, 1955, pp. 145–173. Polytechnic Institute of Brooklyn, Brooklyn, N. Y., 1956.

This paper applies the theory of bounded power series developed by Carathéodory [Math. Ann. **64** (1907), 95–115; Rend. Circ. Mat. Palermo **32** (1911), 193–217] and Schur [J. Reine Angew. Math. **147** (1917), 205–232; **148** (1918), 122–145] to the study of positive real functions in the sense of Brune [J. Math. Phys. **10** (1931), 191–236]. Various network implications are brought out.

R. J. Duffin (Pittsburgh, Pa.).

Bunimovich, V. I. Fluctuating processes as oscillations with random amplitudes and phases. Translated by Morris D. Friedman, 572 California St., Newtonville 60, Mass., 1956. 41 pp.

Translation of Akad. Nauk SSSR. Žurnal Tehn. Fiz. **19** (1949), 1231–1259; MR **14**, 995.

★ **Trakhtenbrot, B. A. Synthesis of non-iterated circuits.** Translated by Morris D. Friedman, 572 California St., Newtonville 60, Mass., 1956. 6 pp.

Translated from Dokl. Akad. Nauk SSSR (N.S.) **103** (1955), 973–976. The original Russian article was reviewed in MR **18**, 860.

★ **Lupanov, O. B. On the possibilities of designing circuits out of various elements.** Translated by Morris D. Friedman, 572 California St., Newtonville 60, Mass., 1956. 5 pp.

Translated from Dokl. Akad. Nauk SSSR (N.S.) **103** (1955), 561–563. The original Russian article was reviewed in MR **18**, 850.

★ **Povarov, G. N. To the study of symmetric Boolean functions from the relay switching circuit theory viewpoint.** Translated by Morris D. Friedman, 572 California St., Newtonville 60, Mass., 1956. 5 pp.

Translated from Dokl. Akad. Nauk SSSR (N.S.) **104** (1955), 183–185. The original Russian article was reviewed in MR **18**, 860.

See also: Ordinary Differential Equations: Gumowski. **Statistical Thermodynamics and Mechanics:** Faïn. **Structure of Matter:** Groschwitz and Siebertz. **Fluid Mechanics, Acoustics:** Stewartson. **Classical Thermodynamics, Heat Transfer:** Mascarenhas. **Quantum Mechanics:** Brillouin; van der Spuy; Glaser and Jakšić.

Classical Thermodynamics, Heat Transfer

Camia, Frédéric. Équations fondamentales des courbes balistiques de la chaleur (et de la diffusion) dans un mur homogène fini. C. R. Acad. Sci. Paris **244** (1957), 1163–1166.

Cinq solutions de l'équation de propagation de la chaleur sont construites, ces solutions correspondent à une propagation rectiligne dans un mur homogène d'épaisseur fini. *H. Cabannes* (Marseille).

Camia, Frédéric. Propagation d'un choc thermique dans un mur homogène fini: passage du maximum. C. R. Acad. Sci. Paris **244** (1957), 2293–2296.

L'auteur utilise les solutions de l'équation de propagation de la chaleur pour étudier la propagation du maximum de température: déplacement de ce maximum au cours du temps et variation de sa grandeur. Deux cas particuliers sont envisagés: 1) mur thermiquement isolé sur une face et à température constante sur l'autre face, 2) mur thermiquement isolé sur les deux faces. *H. Cabannes.*

Mascarenhas, S. Thermodynamical theory of thermal conduction of dielectrics under electric fields. Nuovo Cimento (10) **5** (1957), 1118–1121.

The abnormal heat transmission of a fluid dielectric under electrical fields can be conveniently described as a cross-effect between thermal and electrical forces. The author studies the case of low fields for a plane cell when convection effects are absent. He is also led to the converse effect, i.e., the variation of the electrical conductivity with the thermal force. *Author's summary.*

Mihaĭlov, Yu. A. Application of the system of differential equations of heat- and mass-exchange to the process of convection drying. Latvijas PSR Zinātņu Akad. Vēstis **1956**, no. 12(112), 107–118. (Russian)

Datsev, A. B. On the two-dimensional multi-layer problem of heat conduction. Translated by Morris D. Friedman, 572 California St., Newtonville 60, Mass., 1956. 7 pp.

Translated from the Russian of the article in Dokl. Akad. Nauk SSSR (N.S.) **101** (1955), no. 5, 813–816.

Datsev, A. B. On the three-dimensional multilayer heat conduction problem. Translated by Morris D. Friedman, 572 California St., Newtonville 60, Mass., 1956. 5 pp.

Translated from the Russian of the article in Dokl. Akad. Nauk SSSR N.S. **101** (1955), no. 6, 1019–21.

Datsev, A. B. On the two-dimensional Stefan problem. Translated by Morris D. Friedman, 572 California St., Newtonville 60, Mass., 1956. 7 pp.

Translated from the Russian of the article in Dokl. Akad. Nauk SSSR N.S. **101** (1955), no. 3, 441–444.

See also: Partial Differential Equations: Séminaire Schwartz; Šil'krut; Kolodner. **Elasticity, Plasticity:** Hieke; Chu; Tind and Li; Mirzadžanzade and Abbasov.

Quantum Mechanics

★ **Bohm, David. Causality and chance in modern physics. Foreword by Louis de Broglie.** D. Van Nostrand Company, Inc., Princeton, N.J.-New York, 1957. xi+170 pp. $5.00.

The book contains a thorough, almost pedantic analysis of the concepts of causality and chance. Its language is non-technical although its full understanding (especially of the last 3 chapters) requires a knowledge of modern physics. There are 5 chapters, the first two dealing with causality and chance in natural law and in classical physics. In the third chapter the basic concepts of the quantum theory as usually accepted are introduced and criticized. According to the author, the views of Bohr and Heisenberg lead to the conclusion that the uncertain and incomplete character of the knowledge that experiment gives us about what really happens in microphysics is the result of real indeterminacy of the physical state. The author contrasts with this view his own, de Broglie's and others. According to them (as explained more fully in chapter IV), there should exist below the level of quantum theory, a subquantum level, the laws of which may be entirely different from those on the quantum level. These — subquantum level laws — would apply for very great energies and for distances smaller then 10^{-13} cm and they would have as consequence the ordinary laws of quantum-mechanics. In chapter IV the author sketches (in general terms) his trials leading to such a subquantum theory. The last chapter contains general philosophical ideas mentioned and summarized less explicity before. In the author's own words: (page 100) "As an alternative to the positivist procedure of assigning reality only to that which we now know how to observe, we are adopting ··· a point of view which we believe corresponds more closely to the conclusions that can be drawn from general experience in actual scientific reasearch. In this point of view, we assume that the world as a whole is objectively real, and that, as far we know, it has a precisely analysable structure of unlimited complicity ···. We should, however, never expect to obtain a complete theory of this structure ···. Thus with regard to the current formulation of the quantum theory, we are led to criticize assumptions such as those of Heisenberg and Bohr, that the indeterminacy principle and the restriction to complementary pairs of concepts will persist no matter how far physics may progress into new domains."

Few physicists would agree with everything that was said in this book; but almost everyone would agree that the author presented his case with clairty and intelligence. *L. Infeld* (Warsaw).

★ **Kramers, H. A. Quantum mechanics.** North-Holland Publishing Company, Amsterdam; Interscience Publishers Inc., New York, 1957. xvi+496 pp. $12.50.

The first volume of Eucken and Wolf's "Hand- und Jahrbuch der chemischen Physik" [Akademische Verlagsgesellschaft, Leipzig, 1933] was devoted to a representation of "the theoretical foundations of our present knowledge of the structure and properties of atomic systems". The author, the late H. A. Kramers, divided the subject into two parts, one dealing with the foundations of quantum theory and one presenting the quantum theory of the electron and of radiation. The whole volume, however, has great unity and it constitutes one of the very best presentations of quantum theory (nuclear phenomena excluded) ever published. One of the attractive features is that the author went at all stages into an explicit discussion of the problems of interpretation and of relation to classical theory which are so delicate but form an essential part of the subject, and of which he had an excellent knowledge. Another fascinating aspect, which is now not only of scientific but also of historical importance, is the discussion of the interaction between electrons and the radiation field, which Kramers had brought at the time of publication of his book to its most advanced prewar state of development by recognizing the necessity of what is now called mass renormalization.

The fact that this masterly book was published as a volume of the Handbook of Chemical Physics, and perhaps also in later years the fact that the book was written in German, account for the lack of familarity of many physicists of the younger generation with its contents. It must therefore be considered as an eminent service that D. ter Haar, a former student of Kramers, agreed to translate Kramers' contribution to the "Hand- und Jahrbuch" into English and that the North-Holland Publishing Company has published the translation in the form of a text-book. The translation is highly literal, a fact which is noticeable throughout the text. According to the translator's preface this results from a deliberate choice. In the reviewer's opinion the choice is a sound one in view of the subtle and delicate nature of many discussions. The translator has added references to modern literature and has modified certain more obscure passages.

Part one of the volume, entitled "The Foundations of Quantum Theory", deals with the non-relativistic quantum theory of spinless particles. After chapters on free particles, bound particles and many-body systems, transformation theory, embodying the theory of operators, is analyzed and approximation methods are treated. Remarkably enough the Wentzel-Kramers-Brillouin method is not considered. It is mentioned very briefly, in small print, at the end of part two in the middle of a discussion of the radiation field. Part one has also been published by the same publisher as a separate volume.

Part two is entitled "Quantum Theory of the Electron and of Radiation". It opens with a chapter devoted to the electron as spin one-half particle, non-relativistically first and then according to Dirac. The next chapter discusses the exclusion principle and gives the multiplet theory of atomic systems. The third and last chapter studies the radiation field and its interaction with electrons.

L. Van Hove (Utrecht).

★ **Mandl, F. Quantum mechanics. 2nd ed.** Academic Press Inc., New York; Butterworths Scientific Publications, London, 1957. x+267 pp. $6.50.

The first edition was reviewed in MR **16**, 1078. In this edition, two new topics have been included: a section on partial wave analysis in scattering theory and a chapter on the Dirac equation. The first is important in atomic, nuclear and high-energy physics; the second contains a discussion of the relativistic invariance of the theory which underlies the construction of interactions in beta-decay and field theories generally. Some smaller changes and corrections are also included.

Roussopoulos, Paul. Sur la théorie des processus non stationnaires en mécanique quantique. C. R. Acad. Sci. Paris **244** (1957), 1732–1734.

L'auteur donne une représentation intégrale complexe

pour la solution retardée de l'équation de Schroedinger en représentation d'interaction. Cette solution permet de discuter les conditions dans lesquelles les résultats de Lippmann-Schwinger et d'Arnous-Zienau sont valables.
D. Rivier (Lausanne).

Roussopoulos, Paul; et Vardalakis, Georges. Sur les phénomènes d'affaiblissement et la largeur naturelle des raies spectrales en électrodynamique quantique. C. R. Acad. Sci. Paris **244** (1957), 1888–1890.

Les auteurs appliquent la solution trouvée dans l'oeuvre analysée ci-dessus pour l'équation de Schroedinger en représentation d'interaction au problème de la vie moyenne des états excités d'un atome et de la largeur naturelle des raies spectrales. En utilisant une approximation de Born faisant intervenir complètement les termes du 2e ordre, ils obtiennent la solution du problème limitée à cette approximation. *D. Rivier.*

Winogradzki, J. Spineurs du second rang à composantes invariantes et formalisme spinoriel incluant les parités. J. Phys. Radium (8) **18** (1957), 387–394.

Writing the transformation formula of 4-component Dirac spinors as $\Psi=S\Psi'$, the author defines 4 kinds of spinors of second order N, L, P, and M, with 16 components each. The transformation formula for N is $N=\varepsilon SN'S^{-1}$, where ε depends on the parity. The transformation formulas for L, P, and M are analogous. All invariant spinors of second order are determined by direct solution of the equations $NS=\varepsilon SN$, etc.
B. L. van der Waerden (Zurich).

Bodiou, G. Une forme spinorielle des équations de l'électromagnétisme. J. Phys. Radium (8) **18** (1957), 169–172.

The author uses the known correspondence between self-dual antisymmetric tensors in Minkowski space-time and a pair of two component spinors to write the Maxwell equations in terms of this pair of spinors. The arbitrariness in the determination of each of the two component spinors is used to simplify the resulting equations. When no currents are present the simplified equations are shown to be equivalent to a pair of Dirac equations for particles of zero mass. *A. H. Taub* (Urbana, Ill.).

Englert, François. Application de la théorie des groupes au calcul du couplage spin-orbite dans les cristaux. Acad. Roy. Belg. Bull. Cl. Sci. (5) **43** (1957), 273–283.

★ **Bogoliubov, N. N. On a new form of adiabatic perturbation theory in the problem of particle interaction with a quantum field.** Translated by Morris D. Friedman, 572 California St., Newtonville 60, Mass., 1956. 25 pp.
Translated from the Russian of the article in Ukrain. Mat. Ž. **2** (1950), no. 2, 3–24; MR **13**, 194.

Brillouin, L. Interactions d'ondes et électrons et remarques sur la supraconductivité. J. Phys. Radium (8) **18** (1957), 331–336.

The author reviews in a physical setting the wave-mechanical problem of interaction between an electron and electromagnetic waves or elastic waves in a conductor, the mathematical formulation of which is found in his joint work with M. Parodi [Propagation des ondes dans les milieux périodiques, Masson, Paris, 1956, ch. XIV; MR **18**, 170]. He advances arguments to show that the theory of electron capture, which has been suggested as a basis for explaining the phenomenon of superconductivity, is in its present state unable to provide such an explanation. *R. N. Goss* (San Diego, Calif.).

Hubbard, J. The description of collective motions in terms of many-body perturbation theory. Proc. Roy. Soc. London. Ser. A. **240** (1957), 539–560.

The diagram method, originally applied by Dyson to quantum electrodynamics is used to cast many-body perturbation theory into a convenient form (linked cluster expansion), and to calculate the energy shift due to a given interaction. The method is equivalent to that of Goldstone [same Proc. **239** (1957), 267–279; MR **18**, 975], though it differs from it in a number of points; the paper is also in some respects more detailed. The case of a uniform gas permits certain simplifications which lead to the introduction of a modified interaction. Using this interaction the energy shift can be computed via a series which converges faster than the original perturbation series. This approach is closely related to the dielectric theory of electron plasma oscillations. *F. Rohrlich.*

Halfin, L. A. Physical invariance of quantization. Vestnik Leningrad. Univ. **11** (1956), no. 22, 12–17. (Russian)

It is shown that if the Lagrangian of a system is changed by the addition of a term which does not change the equations of motion, so that thereby the Hamiltonian is changed, the Schroedinger equation can be brought back to its original form by multiplying the wave function by a factor of the form $e^{i\phi}$ where ϕ is a real function. A similar situation holds in the case of a transformation group under which the "classical" equations of motion are invariant. *N. Rosen* (Haifa).

Sokolov, A. A.; Ivanenko, D. D.; and Ternov, I. M. On the excitation of macroscopic oscillations by quantum fluctuations. Dokl. Akad. Nauk SSSR (N.S.) **111** (1956), 334–337. (Russian)

It is shown that a high energy electron in a magnetic field tends to move so that its rotational motion is described by classical mechanics, while its radial oscillations with macroscopic amplitude (representing a kind of "macro-atom") obey the laws of quantum mechanics. The uncertainty relations for this "macro-atom" are discussed. *N. Rosen* (Haifa).

Bocchieri, P.; and Loinger, A. Quantum recurrence theorem. Phys. Rev. (2) **107** (1957), 337–338.

A recurrence theorem is proved, which is the quantum analog of the recurrence theorem of Poincaré. Some statistical consequences of the theorem are stressed.
Authors' summary.

Kahana, S.; Polkinghorne, J. C.; and Martin, J. L. Cauchy's problem in quantum field theory. Proc. Roy. Soc. London. Ser. A. **241** (1957), 408–413.

The relation between the classical and quantum versions of field theories is still relatively unexplored. This paper deals with the question of interpreting Cauchy's problem for the case of a field theory linear in the field amplitudes. For such theories it is known [J. Hadamard, Le problème de Cauchy et les équations aux dérivées partielles linéaires hyperboliques, Hermann, Paris, 1932] that it is not in general possible to specify the values of a field variable $\phi(x)$ on two different space-like surfaces such that $\phi(x)$ satisfies the classical field

equations and the end conditions. The quantum analogue is $\langle\phi'', \sigma''|\phi', \sigma'\rangle=0$ if ϕ'' on σ'' and ϕ' on σ' are inaccessible in the classical sense. The authors evaluate this matrix element using Feynman's sums over paths, and the equation is verified. If ϕ' and ϕ'' are accessible, then it is shown that the classical solution is not unique, and that there is a corresponding ambiguity in the matrix elements of operators. No definite results are reported for the non-linear case. *C. A. Hurst* (Adelaide).

Murota, Toshiyuki; Ueda, Akira; and Tanaka, Hajime. The creation of an electron pair by a fast charged particle. Progr. Theoret. Phys. **16** (1956), 482–496.

Earlier calculations of the cross section for electron pair production by fast charged particles [see, e.g., H. J. Bhabha, Proc. Roy. Soc. London. Ser. A. **152** (1935), 559–586] have been based on the use of the Weiszaecker-Williams treatment of the virtual photon field accompanying high energy charged particles. Recently discrepancies between the results thus obtained and experimental data have been reported. In order to decide whether these discrepancies are due to a failure of the theory itself or to the use of unjustifiable simplifications the authors use the Feynman-Dyson method to calculate the differential cross section of the process. The main contribution to the total cross section is then shown to be due to transitions where the spin of the incident particle does not flip and where the direction of polarization of the virtual photon is transverse. It is further shown that the energy spectrum of the virtual transverse photons coincides with the classical spectrum if the energy transfer from the incident particle to the created pair is small. These conclusions are based on the possibility of separating the contributions to the cross section from virtual photons of different directions of polarization. The results obtained confirm the applicability of the Weiszaecker-Williams method to the problem considered. The cause of the discrepancy between theory and experiment is not explained. *E. Gora* (Providence, R.I.).

Murota, Toshiyuki; and Ueda, Akira. On the foundation and the applicability of Williams-Weizsäcker method. Progr. Theoret. Phys. **16** (1956), 497–506.

In the paper reviewed above it is shown that the applicability of the Weiszaecker-Williams method to pair production by fast particles was essentially due to the possibility of separating the contributions to the cross section from virtual photons of different directions of polarization. A generalization of this conclusion to fields of virtual particles other than photons is now proposed. It is shown that the methods of Weiszaecker-Williams and of Feynman-Dyson are equivalent if the condition $\varepsilon\mu/E_1 m \ll 1$ is fulfilled, where ε is the energy transferred from the incident particle, E_1 and μ are energy and mass of the incident particle, and m is the mass of the created particles. However, it is also shown that the Weiszaecker-Williams method gives the correct order of magnitude for the contributions from regions where this condition does not hold. *E. Gora* (Providence, R.I.).

Sokolov, A. A.; and Tsytovich, V. N. The theory of the electron field mass in the presence of an external medium. Soviet Physics. JETP **3** (1956), 94–97.

To calculate the difference between the value of the electron mass in vacuo and in a medium characterized by its dielectric constant and its magnetic permeability the field equations of quantum electrodynamics are solved in second approximation for the emission and absorption of a photon in the medium. A convergent expression is obtained for this difference and for the corresponding level shift in a hydrogen atom. This shift is much smaller than the Lamb shift, but under normal pressure and temperature larger than the shift due to the interaction between the electron and the radiation field. *E. Gora.*

Steuckelberg, E. C. G.; et Petermann, A. La normalisation des constantes dans la theorie des quanta. Helv. Phys. Acta **26** (1953), 499–520.

Dans ce travail les auteurs se proposent de donner un fondement mathématique au problème de la normalisation des constantes apparaissant dans la matrice S telle qu'elle a été définie dans deux publications antérieures [E. C. G. Stueckelberg et D. Rivier, mêmes Acta **23** (1950), 236–239; E. C. G. Stueckelberg et T. A. Green, ibid. **24** (1951), 153–174; MR **11**, 763; **13**, 191]. Selon celles-là la matrice S s'obtient par construction causale et unitaire à partir de certaines actions locales hermitiennes et invariantes qui sont les correspondants quantiques (nombres q) des termes d'interaction dans une fonction de Lagrange classique (nombre c) décrivant un système de champs interagissant entre eux. Cette fonction de Lagrange classique est notamment invariante par rapport à 3 groupes de transformations à 1 paramètre chacun, les 3 groupes de jauge (métrique, électromagnétique et mésonique). La matrice S ainsi construite contient des constantes arbitraires dont l'origine est la présence dans ses termes de produits de distributions de Dirac et de ses dérivées. (La présence de constantes arbitraires au lieu d'infinités à renormaliser constitue la caractéristique essentielle du formalisme de Steuckelberg). Après avoir montré qu'une variation des constantes arbitraires revient à faire varier les constantes physiques (masses, constantes de couplages, etc.) du système de champs. les auteurs montrent que l'invariance de la matrice S par rapport aux 3 groupes de jauge entraîne les conditions qui déterminent les constantes arbitraires à des termes ou à des facteurs "non physiques" près. *D. Rivier.*

Hamaguchi, Minoru. The generalization of Stueckelberg formalism in the theory of quantized field. Progr. Theoret. Phys. **11** (1954), 461–475.

L'auteur propose une nouvelle méthode pour surmonter les difficultés de "divergences" en théorie des champs quantifiés. Cette méthode combine l'idée d'interaction non locale à une autre idée due à Stueckelberg [Phys. Rev. (2) **81** (1951), 130–133; MR **13**, 191] et selon laquelle, pour éviter les difficultés de divergences provenant des discontinuités spatiotemporelles créées par les surfaces d'observation, il faut introduire des observations dans des régions "diffuses" de l'espace-temps. La réalisation du programme utilise une "extension spatio-temporelle $A[x_\mu(\tau)]$" dont la représentation de Fourier $A[k_\mu]$ vient se substituer à la traditionnelle distribution de Dirac $\delta(k_\mu)$ correspondant à un sommet dans des éléments de la matrice S (représentés dans l'espace des moments k). Après avoir donné une interprétation cinématique du modèle ainsi construit, puis une correspondance avec la théorie à interaction non locale de Paris et Uhlenbeck [ibid. **79** (1950), 145–165; MR **12**, 227] et avec celle de Foldy [ibid. **87** (1952), 688–693; MR **14**, 226], l'auteur applique sa théorie 1) à l'électrodynamique quantique en évaluant la self-énergie de l'électron et à la polarisation du vide

(les résultats obtenus (au 2e ordre) coïncident avec ceux de Feynman [ibid. **76** (1949), 769–789; MR **11**, 795]), 2) au problème du moment magnétique du nucléon.

D. Rivier (Lausanne).

Enz, C. P. Wechselwirkungskräfte und Renormalisation in der Photonpaar-Theorie. Nuovo Cimento (10) **3** (1956), supplemento, 363–417.

The paper discusses the interaction of finite heavy charged spinless particles with photons with no external field in non-relativistic approximation. This is a simplified quantum electrodynamics designed to be used for answering basic questions about renormalization. The author shows 1) that mass renormalization is essential in order to maintain the validity of the classical equations of motion and is connected with the Coulomb self-energy rather than the interaction with the photon field, 2) that when the coupling constant is positive no charge renormalization results from any of the various renormalization prescriptions, 3) that when the coupling constant is negative there is difficulty in defining the vacuum, yet the renormalized correction to Coulomb's law and to the interaction cross-section remains meaningful. *A. J. Coleman.*

Wong, David Y. Dispersion relation for nonrelativistic particles. Phys. Rev. (2) **107** (1957), 302–306.

It is shown that if the wave function of a nonrelativistic particle satisfies the Schrödinger equation with a velocity-independent potential, then its scattering amplitudes (and the S matrix in general) satisfy the same dispersion formulas as those derived for the scattering of light.

From the author's summary.

Tobocman, W. Many-body perturbation theory. Phys. Rev. (2) **107** (1957), 203–208.

The usual perturbation treatment of the many-body problem is modified by eliminating the two-body interaction and introducing in its place a t matrix. The Watson, Brueckner, and Bethe treatment of the many-body problem is shown to be a special case of the modified perturbation treatment. *From the author's summary.*

Gulmanelli, P.; and Montaldi, E. Gravitational forces and quantum field theory. Nuovo Cimento (10) **5** (1957), 1716–1721.

The Gamow-Teller hypothesis, according to which an exchange of neutrino pairs could give rise to a gravitational type potential between nucleons, is re-examined. By introducing a new boson field it is shown that this guess cannot be valid at least in the frame of the usual linear field theories. *Author's summary.*

van der Spuy, E. Investigation of the states of an electron in a proposed electromagnetic field. Nuovo Cimento (10) **4** (1956), 1349–1363.

A treatment, partly qualitative, partly wave-mechanical, is given of the possible determination of the spin moment of an electron by a Rabi type of experiment in which the electron passes through two inhomogeneous magnetic fields separated spatially by a homogeneous magnetic field together with a radiofrequency electromagnetic field. *C. Strachan* (Aberdeen).

Glaser, V.; and Jakšić, B. Electromagnetic properties of particles with spin. Nuovo Cimento (10) **5** (1957), 1197–1202.

The scattering of a relativistic electron on particles with some electromagnetic structure and spin up to $s=\frac{1}{2}$ is investigated, and the cross-sections, in the first Born approximation for the electromagnetic field, are calculated. A possible estimation of the proton radius is given on the basis of quite general arguments.

Author's summary.

Brown, G. E.; and De Dominicis, C. T. Direct interaction and nuclear dispersion theory. Proc. Phys. Soc. Sect. A. **70** (1957), 686–689.

The problem of inelastic scattering of several MeV nucleons is formulated in nuclear dispersion theory.

Weisskopf, Victor F. Nuclear physics. Rev. Mod. Phys. **29** (1957), 174–181.

Nucleon-nucleon scattering cannot be explained by a well-defined potential. The phase shifts deduced from scattering experiments cannot be used for nucleon-nucleon scattering within the nucleus since the separation d of scattering centres may be much less than the wavelength of the nucleon. If r be the radius of the scatterer, say the measured extension of the proton, then $d \ll 2r$ or $d \ll 2r$ may justify the Brueckner or the one-meson field (Teller) approach. Important hints are the universal density of nuclear matter and the shell model successes (independent particles), also the many non-spherical nuclei and the nuclear reaction data. Here the independent particle approach is used with a potential energy well whose depth depends on the state of the particle, e.g. quadratically on the momentum. Giant resonances and moments of inertia of deformed nuclei may confirm the deduced effective mass. A three stage description of nuclear reactions is given in some detail, the independent particle stage (particle interacts with potential V_1+iV_2, optical model), the compound system stage (particle removed from entrance channel), and final stage of separation of reaction products. *C. Strachan* (Aberdeen).

Golden, Sidney. Statistical theory of many-electron systems. General considerations pertaining to the Thomas-Fermi theory. Phys. Rev. (2) **105** (1957), 604–615.

A generalization of the statistical theory of many-electron systems of Thomas and Fermi is achieved by the use of the density matrix. First, a procedure for the expansion of an operator given as a function of the classical Hamiltonian is developed which permits an expansion of the single-particle density matrix in terms of a power series in $\hbar$. The zeroth order terms in this expansion correspond to the Thomas-Fermi theory. This expansion is then used to derive an expression for the diagonal elements of the single-particle density matrix in second order approximation. This expression is utilized to calculate the free-energy function and the total energy of the system. Some of the second-order terms thus obtained are identified as corresponding to exchange energy and to the Weizsäcker inhomogeneity correction for the kinetic energy; their presence follows directly from the expansion procedure. The formalism is used to calculate the energies of the atomic states H(^2S), He(^1S), He(^3S), C^{++}(^1S), and O(^1S). The disparity between calculated and experimental values is about 6% for He(^3S), and less the 2.5% for the other atomic states considered. *E. Gora.*

Golden, Sidney. Statistical theory of many-electron systems. Discrete bases of representation. Phys. Rev. (2) **107** (1957), 1283–1290.

In the paper reviewed above a formulation of the sta-

tistical theory of many-electron systems based on an evaluation of the density matrix in terms of the momentum eigenfunctions of the continuous spectrum has been given. In the present paper an extension of this formulation is achieved by the evaluation of the density matrix in terms of a basis corresponding to a discrete spectrum of eigenvalues. This evaluation is performed in a quasiclassical approximation where a projection operator introduced by Per-Olov Löwdin [Phys. Rev. (2) **97** (1955), 1474–1489, 1490–1508, 1509–1520; MR **16**, 983] selects the portion of the density matrix corresponding to a combination of a finite number of one-particle wavefunctions. The desired quantities are then approached by finite products of exponential factors in a manner that permits any given approximation to contain all the terms of the previous approximation. The first order approximation is employed to calculate the energies of the ground ^1S state of helium and the three lowest states of Li^+ and Be^{++}. The discrepancy between calculated and experimental values is less than one per cent. In developing the formalism, the quantity $\exp(zH)$, where H is the many-particle Hamiltonian and z is a complex number, is represented as an infinite product of exponential factors, each of which depends upon a portion of the Hamiltonian alone. If one puts $z=-1/kT$, one obtains from this representation an explicit expansion procedure for the partition function solely in terms of the exponentials of the portions of the Hamiltonian. *E. Gora* (Providence, R.I.).

Ladik, J.; and Csukás, A. Determination of the magnetic interaction in the H_2 molecule due to the motion of two electrons. Acta Phys. Acad. Sci. Hungar. **6** (1957), 381–397. (Russian summary)

The magnetic interaction of the two electrons in the ground state of the hydrogen molecule, arising from their motion, is calculated by using the approximate wave function of S. C. Wang [Phys. Rev. (2) **31** (1928), 579–586] and is found to be $8.24.10^{-4}$ e.v. Calculations of two molecular integrals are given. *N. Rosen* (Haifa).

Berencz, F. Die Berechnungen des Grundzustandes des Wasserstoffmoleküls auf Grund des Variationsverfahrens. Acta Phys. Acad. Sci. Hungar. **6** (1957), 423–441. (Russian summary)

The ground state of the hydrogen molecule is calculated by the variational method with a wave function having a form related to that of the Hund-Mulliken function, but containing two variational parameters. A more accurate calculation is also carried out with a similar function multiplied by a correlation factor $(1+pr_{12})$, where p is a parameter and r_{12} is the interelectron distance. In the latter case the dissociation energy differs by about 10% from the empirical value. Expressions are given for a number of molecular integrals. *N. Rosen* (Haifa).

Grard, Fernand. Sur l'excitation et l'ionisation des atomes par émission α. C. R. Acad. Sci. Paris **243** (1956), 777–780.

The author, using the method of H. M. Schwartz [Phys. Rev. (2) **100** (1955), 135–137] confirms the calculations of J. S. Levinger [ibid. **90** (1953), 11–25] rather than those of A. Migdal [Acad. Sci. USSR. J. Phys. **4** (1941), 449–453] as Schwartz had done. *N. Rosen.*

Marx, G. Relativistic effects in heavy nuclei. Nuclear Phys. **1** (1956), 660–669.

A relativistic generalization of the classical scalar field theory of nuclear forces of Johnson and Teller [Phys. Rev. (2) **98** (1955), 783–787] is proposed. These authors concluded that in such a theory saturation and the neutron excess in heavy nuclei could only be explained if a velocity dependent term were added to the Hamiltonian. Such a term follows automatically from the proposed relativistic generalization. *E. Gora* (Providence, R.I.).

Gilvarry, J. J. Thermodynamic functions on the generalized Thomas-Fermi-Dirac theory. Phys. Rev. (2) **107** (1957), 33–40.

The generalization of the Thomas-Fermi-Dirac atom model to include a temperature perturbation of first order is considered, as formulated from a variational procedure by Umeda and Tomishima [J. Phys. Soc. Japan **8** (1953), 360–364]. *From the author's summary.*

Frahn, W. E.; and Lemmer, R. H. Velocity-dependent nuclear interaction. Nuovo Cimento (10) **5** (1957), 1564–1572.

The single particle wave equation describing the motion of nucleons in nuclei, as derived from the nuclear many-body problem, is of a non-local form in coordinate space. It is shown, that in the effective mass approximation, this equation reduces to a velocity-dependent Schrödinger equation, which contains a spatially variable effective nucleon mass in a properly symmetrized kinetic energy operator. The eigenvalue problem is treated for the special case of the infinite harmonic oscillator potential as the local part of the interaction. *Authors' summary.*

Power, E. A.; and Zienau, S. On the radiative contributions to the Van der Waals force. Nuovo Cimento (10) **6** (1957), 7–17.

The effect of the electromagnetic interaction between two neutral atoms is important for large separations where the London potential proportional to R^{-6} is altered and becomes the Casimir-Polder potential varying as R^{-7}. It is shown that the interaction energy, for all separations, may be computed in a relatively simple manner by quantum electrodynamics with the aid of a reduced Hamiltonian involving transverse quanta alone. This Hamiltonian function is justified by an appeal to the dipole approximation in the Lagrangian. *Authors' summary.*

Kaempffer, F. A.; and Kennedy, Edith M. Numerical analysis of a unitary particle model. Canad. J. Phys. **35** (1957), 48–54.

A set of non-linear equations which can serve as the basis for a classical unitary theory of neutral particles is investigated. Approximate expressions are obtained for particle-like solutions, depending on a parameter, which is then chosen to minimize the energy. A spectrum of masses is obtained of which the first three are in ratios not very different from those of known heavy neutral particles. Some limitations of the model are discussed.

N. Rosen (Haifa).

Aržanyh, I. S. On chain systems of a meson field. Dokl. Akad. Nauk SSSR (N.S.) **110** (1956), 351–354. (Russian)

To describe a meson field having a spectrum of masses for a given velocity or a spectrum of velocities for a given mass, systems of equations are set up. Recursion relations among the solutions are derived. *N. Rosen* Haifa).

Gell-Mann, Murray. Model of the strong couplings. Phys. Rev. (2) **106** (1957), 1296–1300.

A tentative conclusion from experiment, chiefly photoproduction, is that baryons are not coupled to K mesons as strongly as to pions. Hypofragments must then be bound, predominately, by pions. Simple calculations indicate that this requires Λ and Σ particles to have the same parity.

Starting from these remarks a theoretical scheme is proposed in which, if the (moderately strong) K couplings are turned off, a higher symmetry is revealed: the baryons degenerate into four similar doublets, of identical mass and identical (very strong) π interactions. Some consequences of this scheme are derived by applying perturbation theory to K interactions but treating π interactions exactly. The difficulty of understanding the $\Sigma^{\pm}$ mass difference on this scheme is noted. *J. C. Taylor.*

Wentzel, G. A hyperon model. Helv. Phys. Acta **30** (1957), 135–142.

It is suggested that Λ° is the only elementary particle among the hyperons, while other hyperons represent bound states of Λ°, nucleons and K-mesons. A qualitative theory is worked out along these ideas by using the mathematical treatment of an earlier paper [G. Wentzel, Helv. Phys. Acta **13** (1940), 269–308]. *S. N. Gupta.*

Borgardt, A. Non-linear meson field of a nucleon at rest. Dokl. Akad. Nauk SSSR (N.S.) **109** (1956), 1107–1108. (Russian)

The equation

$$\beta^4\partial_4\psi+k_0\psi+(\lambda/k_0)\psi^+\beta\psi R\beta\psi=0$$

representing a meson field interacting with a nucleon at rest where β^4 are Kemmer matrices and R is an operator whose form depends on the nature of the interaction, is reduced by assuming spherical symmetry of the source. In the limit $k_0\to 0$, the problem is reduced to a non-linear second order differential equation, whose solution is given in terms of a single quadrature. *A. J. Coleman.*

Rayski, Jerzy. Introduction to the bilocal theory of elementary particles. Acta Phys. Polon. **14** (1955), 337–364. (Russian summary)

Hanus, W.; and Rayski, J. On the mass spectra for bosons. Acta Phys. Polon. **15** (1956), 117–122. (Russian summary)

Within the context of Rayski's bilocal theory [Nuovo Cimento (9) **10** (1953), 1729–1735; Acta Phys. Polon. **13** (1954), 77–88; MR **12**, 766] these papers obtain a mass-spectrum for all elementary particles except photons and gravitons. *A. Salam* (Cambridge, England).

Arnous, E. Etude du nombre de mésons virtuels sur un modèle simple. Nuovo Cimento (10) **5** (1957), 483–490.

The simple model of a scalar neutral meson field interacting with static sources is used to estimate the average number of mesons in the meson cloud around a nucleon. This number is found to be of order 2 for a nucleon radius of order 10^{-13} cm and a coupling constant of order $(\hbar c)^{\frac{1}{2}}$. *L. Van Hove* (Utrecht).

Case, K. M. Reformulation of the Majorana theory of the neutrino. Phys. Rev. (2) **107** (1957), 307–316.

The Majorana theory for a particle with mass is reformulated in terms of a two-component field. In this form it is seen that the theory goes over continuously to the Weyl-type two-component equation as the mass tends to zero. The asymmetries obtained in β and μ decay experiments are shown to imply only that the neutrino mass is small—not that it is zero. Also the asymmetries are shown to be no more implied by the use of a two-component theory than by the use of a four-component one. Two-component theories do imply certain relations between mass, parity nonconservation, and double β decay. That there are asymmetries is due to the fact that the interactions necessary to describe the physical world are not reflection-invariant. *Author's summary.*

Michel, Louis. Weak interactions between "old" particles and beta decay. Rev. Mod. Phys. **29** (1957), 223–230.

Is it possible to explain all the properties of the 'old' particles by adding one interaction ('Fermi' interaction) to gravitational, electromagnetic, and nuclear forces? First is given a detailed summary of the evidence for the definition of the beta-decay interaction and its relation to muon capture and muon decay. The same Fermi coupling can explain all three. However the pion decay, e.g. in the very small ratios of the decay rates involving an electron to those involving a muon, remain a challenge. Other topics, e.g. double beta-decay are mentioned briefly.

C. Strachan (Aberdeen).

Low, F. E. Pi mesodynamics. Rev. Mod. Phys. **29** (1957), 216–222.

Attention is mainly confined to low energy, 'real' (not 'virtual') phenomena for pi mesons, viz. scattering and photoproduction, the last being just mentioned. Experimentally known properties are discussed including the 3-3 resonance in the cross-section for scattering by protons. Theoretically available are the pseudoscalar theory, certainly incomplete, and usable for sum-rules, etc., but not for calculations, then the Chew theory for a nucleon of infinite mass, which is divergent and requires a cut-off but can be used to calculate the 3-3 phase-shift, the resonance shape and photomeson production. This success is discussed by means of equations based on the static theory. These equations are then tentatively derived from more reasonable assumptions than the static theory. Thus, while neither theory allows calculation of the resonance energy, the experimental value of this enables the shape of the resonances and the Watson-Lepore coupling constant to be calculated. In finer details, e.g. with regard to S-waves, the theory has little success. *C. Strachan.*

Yukawa, Hideki. Introductory remarks on meson theory. Rev. Mod. Phys. **29** (1957), 213–215.

Limitations and partial successes of present meson theories and related theories of strange particles receive brief comment from many points of view. The basis of a new theory to describe strong and weak interaction for the same particles is sketched. The known principles and symmetry properties of present-day physics remain, but the field for each type of particle is the product of a function of the usual space-time coordinates and a function of coordinates in an internal world which has the same space-time structure as the external one. Simplifying assumptions produce acceptable strong and weak interactions but do not explain the θ and τ modes of decay of K mesons.

C. Strachan (Aberdeen).

See also: Partial Differential Equations: Stummel. **Numerical Methods:** Kotani, Amemiya, Ishiguro, and Kimura; Ishiguro, Arai, and Sakamoto; Alder and Winther. **Probability:** Harris. **Structure of Matter:** Elliot; Judd, and Runcimann. **Optics, Electromagnetic Theory, Circuits:** Lenz and Scheffels.

Relativity

Gomes, Ruy Luis. Kinematics of rigid bodies in relativity. Gaz. Fis., Lisboa 3 (1956), 99–107. (Portuguese)

M. Born proposed a definition of a moving rigid body in Special Relativity which later was discussed by G. Herglotz and F. Noether, the main result being that there are two types of such motions, one in which the paths of the points of the body are orthogonal trajectories of a system of hyperplanes and the other in which these paths are trajectories of a one parameter group; the number of degrees of freedom for such motions is three and one. The discussion has been extended to General Relativity by G. Salzman and A. H. Taub [Phys. Rev. (2) **95** (1954), 1659–1669; MR **16**, 185]. The present paper arrives essentially at the same conclusions. *G. Y. Rainich.*

Ehlers, Jürgen. Exakte Lösungen der Einstein-Maxwellschen Feldgleichungen für statische Felder. Z. Physik **140** (1955), 394–408.

Einstein's equations with either pure electric or pure magnetic field are equivalent, in the static case, to equations in 3-space. The latter are simplified by conformal transformations and a number of exact solutions obtained for cases of spherical and cylindrical symmetry. As remarked in the paper reviewed below most of these solutions were previously obtained by Datta Majumdar [Phys. Rev. (2) **72** (1947), 390–398; MR **9**, 213].

A. J. Coleman (Toronto).

Ehlers, Jürgen. Beiträge zur Theorie der statischen Vakuumfelder in der klassischen und der erweiterten relativistischen Gravitationstheorie. Z. Physik **143** (1955), 239–248.

The author shows that each solution of the equations $G_{\mu\nu}=-2aV_{,\mu}V_{,\nu}$ in 3-space gives rise rather easily to (i) a Maxwell-Einstein field (ii) a solution of the Jordan-Thiry gravitational field and, if $a\neq 0$, to (iii) an Einstein gravitational field. Hence he derives, quite elegantly, most of the known exact static solutions of Einstein's equations, including the Schwarzschild line-element. Three dimensional spaces satisfying the above equations are characterized by the author's ninth theorem: The Riemannian curvature K of a 2-space is given by $K=-2a[(\partial_n V)^2-\frac{1}{2}g^{rs}\partial_r V\partial_s V]$ where n is the normal to the 2-space. The square bracket is the same as the expression for the normal stress in a classic electrostatic field. *A. J. Coleman.*

Ehlers, Jürgen. Elektrostatische Felder in der erweiterten Gravitationstheorie. Z. Physik **146** (1956), 515–526.

The author of this paper demonstrates again the effectiveness of the method, which he has used in the paper reviewed above for obtaining static solutions of the field equations of the general theory of relativity.

Projective relativity theory is only one of several possible syntheses of gravitational and electromagnetic field theory, but it is a fairly natural extension of general relativity theory and is today perhaps the only important competitor with the "unified" theory. As the author shows, it has the advantage over its rival, that it is mathematically simpler to handle. Exact radially symmetric solutions are obtained which, one might suppose, represent the combined gravitational and electromagnetic field of a charged body. They differ from the Reissner-Nordström solution in taking account of the cosmic variable which, in the projective theory, replaces Einstein's cosmic constant. In the transformed action which the author adopts, the cosmic variable manifests itself as a scalar field coupled to the electromagnetic field and propagating with the velocity of light. The effect of such a field would presumably be very small, though not beyond the limit of experimental detection. *H. S. Green.*

Hönl, H.; und Maue, A.-W. Über das Gravitationsfeld rotierender Massen. Z. Physik **144** (1956), 152–167.

The classic discussion by Thirring [Phys. Z. **19** (1918), 33–39; **22** (1921), 29–30] of the application of the linearized Einstein equations to a sphere rotating with angular velocity ω is here simplified and improved. Thirring found effects of Coriolis type proportional to ω and of centrifugal type proportional to ω^2. The present treatment improves on Thirring by introducing stresses necessary to make $T_{ij}=0$ for $i, j=1, 2, 3$. This results in reducing the centrifugal effect to one-half of that predicted by Thirring, which was already far below the observed value. The calculations are carried through to order ω^2 by an elegant expansion of the g_{ij} in terms of spherical harmonics. An analogy between the resulting equations of motion and those for the motion of a charged particle in an electromagnetic field is used to solve the equations and reveal their physical implications. The paper concludes with a suggested theory of gravitation using a four-vector, which though unsatisfactory in its tensorial behaviour, provides a good qualitative treatment of the phenomena predicted by the linearized Einstein theory to order ω^2.

A. J. Coleman (Toronto).

Takeno, Hyôitirô. On some spherical wave solutions of non-symmetric unified field theories. Tensor (N.S.) **6** (1956), 90–103.

L'auteur considère l'espace-temps statique à symétrie sphérique général,

$$(1)\qquad ds^2=-A(r)dr^2-r^2(d\theta^2+\sin^2\theta d\varphi^2)+C(r)dt^2,$$

calcule à l'aide des équations de Maxwell le tenseur F_{ij} correspondant et définit un tenseur g_{ij} dont la partie symétrique est donnée par le ds^2 (1) (l'auteur suppose ainsi que cette partie donne le champ de gravitation, ce qui semble inexact d'après des résultats récents [cf. Lichnerowicz, Théories relativistes de la gravitation et de l'électromagnétisme, Masson, Paris, 1955; MR **17**, 199; Maurer-Tison, C. R. Acad. Sci. Paris **242** (1956), 1127–1129; MR **17**, 907]) et la partie antisymétrique par:

$$F_{ij}=\tfrac{1}{2}\varepsilon_{ijab}-g^{ia}g^{ab}.$$

Après avoir calculé les Γ_{ij}^{k} associés à les g_{ij} par les formules de la théorie unitaire d'Einstein (et constaté que $\Gamma_i=0$) l'auteur écrit les R_{ij} correspondante et montre que les g_{ij} sont solution approchée des équations de Schrödinger:

$$R_{ij}+\lambda g_{ij}=0$$

$$(R_{ij}+\lambda g_{ij})_{,k}+(R_{jk}+\lambda g_{jk})_{,i}+(R_{ki}+\lambda g_{ki})_{,j}=0$$

si et seulement si le ds^2 (1) est celui de l'espace-temps

de de Sitter de courbure 4λ (cas particulier d'Einstein $\lambda=0$, espace temps de Minkowski).

Les détails de l'interprétation physique sont laissés à l'avenir. *Y. Fourès-Bruhat* (Marseille).

See also: Statistical Thermodynamics and Mechanics: Leruste. **Quantum Mechanics:** Bodiou; Glaser and Jakšić; Marx. **Relativity:** McVittie.

Astronomy

Mihailović, Dobrivoje. Beiträge zur Untersuchung des Zweikörperproblems mit veränderlicher Massensumme. Univ. Beogradu. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. no. 3 (1956), 39 pp. (Serbo-Croatian. German summary)

This paper is the assembly of three papers concerning the same matter previously edited in Serbian [Bull. Soc. Math. Phys. Serbie **4** (1952), no. 3–4, 49–52; **5** (1953), no. 1–2, 67–76, no. 3–4, 93–109; MR **15**, 903, 653, 904] and presents the continuation of the Batirev and Duboshin works. First, one gives the basic equations of the point with the variable mass (Meshtcherskii, Levi-Civita, Gyldèn) and the explanation of the Batirev's paper [Astr. Ž. 26 (1949), 56–59; MR **10**, 577]. The transformations for reducing the Batirev's problem to the classical two-body problem are shown. Further, the general transformations and the relations among the functions $F(t)$, $\varphi(t)$, $M(t)$, $M=m_1+m_2$, are derived and it is shown that the Batirev's transformations are the special cases of these transformations. The qualitative analysis of the forms of the trajectories is given and also the kinematic relations. Using the Batirev's transformations and Milankovic's vector elements [Acad. Serbe. Bull. Acad. Sci. Math. Nat. A. no. **6** (1939), 1–70; Elements of celestial mechanics, 2nd ed., Naucna Knjiga, Beograd, 1955; MR **11**, 407; **17**, 1242] the solutions for the various cases of the motions are obtained. Finally, by means of the assumption that $M=\phi|s|$, $s=m_1m_2$, this problem is reduced to the classical two body problem and the connection with the Radzievskii's three body problem is discussed [Dokl. Akad. Nauk SSSR (N.S.) **91** (1953), 1309–1311; MR **15**, 653]. *D. Rašković* (Belgrade).

Gerstenkorn, Horst. Veränderungen des Erde-Mond-Systems durch Gezeitenreibung in der Vergangenheit bei zeitabhängiger Gravitationskonstante. Z. Astrophys. **42** (1957), 137–155.

The results of an earlier paper of the author [Z. Astrophys. **36** (1955), 245–274; MR **17**, 93] are extended to include the case of gravitational constant f which is inversely proportional to the age of the universe: $f\sim 1/t$. A previous result, according to which in the past the moon was considerably closer to the earth than at present but never inside the stability boundary of Roche, is confirmed. This change in the system of earth-moon, however, has been accomplished (by tidal friction) in the relatively short time of 0.6 to 0.8 milliard years instead of 2.5 milliard years as obtained in the earlier paper. This shortening of the time scale is a direct consequence of a recalculation of the moment of momentum of the tidal friction in terms of the secular accelerations ν and ν' of the moon and the sun under the assumption $f\sim 1/t$. This assumption seems to be in better agreement with the observed value of the quotient $\nu/\nu'\approx 3$ than the assumption $f=$const of the previous paper. Since the time when the distance between the moon and the earth was smallest, the moment of momentum of the system earth-moon has decreased by ca. 2 per cent. From these results the author concludes that originally the moon was an independent celestial body which in the course of geological time has been captured by the earth. *E. Leimanis* (Vancouver, B.C.).

Wilkens, Alexander. Über die Integral-Invarianten der Störungstheorie. Bayer. Akad. Wiss. Math.-Nat. Kl. S.-B. **1955**, 123–173 (1956).

Das allgemeine relative Dreikörperproblem besitzt außer dem Integral der Energie und den drei Flächenintegralen keine weiteren Integrale von bekannter und für alle Zeiten gültiger Form. Es lassen sich jedoch die schon von H. Poincaré untersuchten Integralinvarianten aufstellen, d. h. Konstante, die sich linear aus Integralausdrücken über die gestörten oskulierenden Bahnelemente zusammensetzen. Obwohl diese Konstanten nicht die Bedeutung neuer Integrationskonstanten haben, sondern lediglich Funktionen der bekannten algebraischen Integrale sind, können sie doch als wichtige Kontrollen für die Störungsrechnung bzw. die numerische Integration spezieller Fälle des Dreikörperproblems nützlich sein, ähnlich wie das bekannte Tisserandsche Kriterium, nach dem auch nach dem Durchgang eines Kometen durch das Attraktionsfeld eines störenden Planeten, der eine gründliche Änderung der Bahnelemente verursacht, dennoch eine gewisse Funktion der Bahnelemente fast ungestört bliebt. Verfasser wendet sein Verfahren des systematischen Aufsuchens solcher Invarianten auf verschiedene Fälle der Himmelsmechanik an, u.a. auf das eingeschränkte Dreikörperproblem, in dem diese nützlichen Beziehungen besonders einfache Form annehmen. *K. Stumpff* (Zbl. **70** (1957), 455).

Sultanov, G. F. Double averaging scheme of the type of Gauss. Akad. Nauk Azerbaĭdžan. SSR. Dokl. **12** (1956), 87–89. (Russian. Azerbaijani summary)

L'auteur étudie le problème restreint des trois corps (Soleil, Jupiter, Astéroïde). Soit W_f la fonction perturbatrice; comme dans le cas général on ne peut pas obtenir une solution sous forme finie, l'auteur essaie de résoudre le problème en remplaçant W_f par une fonction doublement moyenne

$$[[W_f]]=\frac{1}{4\pi^2}\int_{w=0}^{2\pi}\int_{\Omega=0}^{2\pi} W_f\, dwd\Omega$$

ce qui lui permet d'obtenir trois intégrales simples du système. En utilisant ces intégrales on peut obtenir le temps t et les deux variables w et Ω par des quadratures. *M. Kiveliovitch* (Paris).

Sklianski, A. L. Sur la question de la classification des cointonations dans le problème généralisé des trois corps. Ukrain. Mat. Ž. **9** (1957), 66–81. (Russian. French summary)

In a generalized problem of three bodies mutually attracting or repulsing under the action of forces with moduli

(1) $m_im_j|f(r_{ij})|$ $(i\neq j;\ \lim_{r\to 0} 2^{k+1}f(r)=-2k<0)$

or with moduli

(2) $g^2m_im_j|\ln r_k/a|$ $(i, j, k=0, 1, 2;\ i\neq j\neq k)$,

the author investigates the motion of that body which does not participate in a binary collision. [Cf. Yu. D. Sokolov, Singular trajectories of a system of free material

points, Akad. Nauk Ukrain. SSR, Kiev, 1951; MR **14**, 910; and A. L. Sklyanskiĭ, Ukrain. Mat. Ž. **6** (1954), 349–362; MR **17**, 198.] In the case of law (1) any binary collision takes place in the invariable plane provided that $f(r)$ satisfies one of the following conditions

$$\lim_{r\to 0} r^{2k+1}f(r)=A,\ \lim_{r\to 0}\frac{f(r)}{\ln r}=B,$$

where A and B are certain constants and $k<1$. For $k=\frac{1}{2}$ a particular case is obtained which has been the subject of investigations of J. Chazy [C. R. Acad. Sci. Paris **168** (1919), 81–83; Ann. Sci. Ecole Norm. Sup. (3) **39** (1922), 29–130] and A. A. Markov [Byull. Astr. Inst. **1** (1926), 159–160; Z. Leningrad. Fiz.-Mat. Obšč. **2** (1929), 81–97].

If $f(r)=1/r^{2k+1}$ $(0<k<1)$, then, as $t\to t_1$, where t_1 in the terminology of Markov denotes an instant of spatial anorthogonal collision between two bodies, the distance of the third body to the invariable plane is of the order

$$(t_1-t)^{(4+2k)/(k+1)}.$$

For $k=\frac{1}{2}$ the law of "ten thirds" of Markov [loc. cit.] is obtained.

Finally, the order of the distance of the third body to the invariable plane is determined and the collisions are classified in the case of the logarithmic law (2) and in the case of law (1) for $k<0$. *E. Leimanis.*

Steffensen, J. F. On the problem of three bodies in the plane. Mat.-Fys. Medd. Danske Vid. Selsk. **31** (1957), no. 3, 18 pp.

Three bodies with finite masses are assumed to move in a plane, subject to Newton's law of gravitation. By the introduction of suitable auxiliary variables the equations of motion are transformed into a system of differential equations of the second degree, permitting expansion of the unknown quantities in powers of the time t, the coefficients of t^p being calculated by means of a set of recurrence formulas. Sufficient conditions for the convergence of the resulting series are given, and the practical working of the method is illustrated by a numerical example. [Cf. also Steffensen, same Medd. **30** (1956), no. 18; MR **18**, 263.] *E. Leimanis* (Vancouver, B.C.).

Mestel, L.; and Spitzer, L., Jr. Star formation in magnetic dust clouds. Monthly Not. Roy. Astr. Soc. **116** (1956), 503–514 (1957).

By applying both the virial theorem [cf. Chandrasekhar and Fermi, Astrophys. J. **118** (1953), 116–141; MR **15**, 168] and the condition for lateral equilibrium of a spiral arm, the authors first show that a strong large-scale magnetic field will prevent star formation as long as the field is frozen into the cloud. The main part of the paper is concerned with how matter may slip across the lines of force and enable star formation. The mechanism proposed is the following: In regions where the hydrogen is principally un-ionized, there will be a frictional coupling between the plasma (consisting of electrons and ions) and the neutral gas. This frictional coupling will contribute to a relatively rapid decay of the field [cf. Piddington, Monthly Not. Roy. Astr. Soc. **114** (1954), 638–650; MR **17**, 691; and Cowling, ibid. **116** (1956), 114–124; MR **18**, 779]; and when the magnetic energy has been reduced in this manner to a small fraction of the gravitational energy, the cloud can break-up into stars. *S. Chandrasekhar.*

Keller, Geoffrey. Astronomical scintillation and atmospheric turbulence. Comments on several recent papers. Astr. Nachr. **283** (1956), 85–86.

L'auteur critique le résultat des recherches de M. Scheffler [Astr. Nach. **282** (1955), 193–205], à savoir que la fonction de corrélation entre le champ électrique et les variations de la phase est indépendante de la hauteur de la couche turbulente. *M. Kiveliovitch* (Paris).

Scheffler, H. Bemerkungen zur Theorie der astronomischen Sicht. Astr. Nachr. **283** (1956), 87–88.

Réponse à la note analysée ci-dessus et justification du résultat obtenu. *M. Kiveliovitch* (Paris).

Piddington, J. H. Solar atmospheric heating by hydromagnetic waves. Monthly Not. Roy. Astr. Soc. **116** (1956), 314–323 (1957).

The role of neutral atoms in the dissipation of magnetic energy in an ionized gas is considered. The charged particles are anchored firmly to moving lines of force while the neutral atoms follow the motions of the lines of force only because of collisions. The fewer these collisions, the greater is the relative velocity of neutral atoms and charged particles and the greater is the dissipation of magnetic energy at the few collisions that actually occur. This mechanism of dissipation is treated mathematically in terms of the equations

$$(1)\quad \nabla^2\mathbf{H}=4\pi\sigma_3\Big\{\frac{\partial\mathbf{H}}{\partial t}-\operatorname{curl}(\mathbf{v}\times\mathbf{H})+\frac{\sigma_2}{\sigma_1}\frac{\partial}{\partial z}\operatorname{curl}\mathbf{H},$$

$$(2)\quad \frac{\partial\mathbf{v}}{\partial t}+(\mathbf{v}-\mathbf{v}')\frac{\eta}{\tau_n}+\frac{V^2}{H_0^2}(\mathbf{H}_0\times\operatorname{curl}\mathbf{H})=0,$$

$$(3)\quad \frac{\partial\mathbf{v}'}{\partial t}=\frac{(\mathbf{v}-\mathbf{v}')}{\tau_n},$$

where $\mathbf{H}$ is the magnetic field having a steady component H_0 along the z-axis and $\mathbf{v}$ and $\mathbf{v}'$ are the fluid velocities of ionized and the neutral components of the gas and σ_1, σ_2 and σ_3 are the three coefficients which the author has introduced in an earlier paper [same Not. **114** (1954), 638–650; MR **17**, 691] η is the ratio of the mass densities of the neutral atoms and ions and τ_n is the collision period (i.e. mean period which elapses before atoms lose their ordered motion relative to the ion plasma). Solutions of equations (1), (2) and (3) are sought which represent the propagation of plane waves in a direction making an angle ψ with the z-axis. If k denotes the wave number and ω the frequency of the waves, then the required dispersion relation is

$$\frac{\omega^2}{k^2}=QV^2S^2+\frac{i}{\varepsilon},$$

where

$$S=\Big[1+\tfrac{1}{2}\sin^2\psi\mp\Big\{\frac{1}{4}\sin^4\psi+\frac{\omega^2\cos^2\psi}{\Omega^2Q^2}\Big\}^{\frac{1}{2}}\Big],$$

$V=H_0/(4\pi\rho)^{\frac{1}{2}}$ is the Alfven velocity, $\varepsilon=4\pi\sigma_3/\omega$, $\Omega/\omega=(\varepsilon\sigma_1V^2)/\sigma_2$ and $Q=\{1+\eta(1+i\omega\tau_n)^{-1}\}^{-1}$. The coefficient of dissipation, $\varkappa$, is given by

$$\varkappa=-\operatorname{Im}(\omega/Q^{\frac{1}{2}}SV).$$

For $\omega\tau_n\ll 1$,

$$\varkappa\approx\frac{\omega^2\tau_n\eta}{2VS(1+\eta)^{\frac{1}{2}}}.$$

The author shows that under certain astrophysical conditions this can be very large: in particular, he attributes the heating of the corona to this mechanism.

S. Chandrasekhar (Williams Bay, Wis.).

★ **McVittie, G. C. General relativity and cosmology.** John Wiley & Sons Inc., New York, 1956. x+198 pp. $9.00.

In the introduction the author stresses the inevitability of taking, implicitly or explicitly, a definite philosophical position in discussing questions of general relativity and does it rather explicitly. He forgoes then an elaborate inductive derivation of the basic equations, using instead analogy with the Newtonian theory. The introduction of the foundations of the theory follows, in general, customary lines but keeps rather close to physical interpretation wherever it is possible, paying attention to units and extensively illustrating the discussion with numerical values. In discussing the classical tests the author marshals an imposing array of observational data but is rather conservative in his conclusions.

Most interest attaches to the remaining portions of the book. Chapters VI and VII are concerned with approximations of the general relativity theory as applied to gas dynamics. One type of approximation corresponds to the assumption ("for the sake of mathematical simplicity") that the coefficients (in the orthogonal case) of the metric differ by small quantities from those of the Minkowski space-time — the smallness being defined in terms of a constant κ in which the coefficients are linear. The other type of approximation corresponds to the expansion of these coefficients in terms of the square of the reciprocal of the speed of light and keeping only the first few terms of the expansion. The author pushes the discussion of these approximations "a little deeper" than it is usually done; he finds that this gives him a better insight into the situation and perhaps more importantly, furnishes, as a by-product, a method for the solution of the equations of classical gas-dynamics. The cosmological term in the Einstein equations significantly enters in the discussion. To the first approximation in the above mentioned constant κ the Newtonian approximation to Einstein's equations gives the hydrodynamic motion of the fluid, the only force being the pressure-gradient; the fluid is assumed to be perfect and the density variable so that the fluid is identified with gas. Taking into account terms in κ^2 introduces self-gravitation of the field.

The last two chapters are devoted to cosmological considerations — they deal with galaxies and other objects outside the solar system. Here the abundance of theoretical possibilities is matched by the comparative paucity of observational data; even when one limits himself, as the author does, to the one special case of the spherical symmetric line-element there are more parameters than can be determined in the present state of information about the Universe. The situation is complicated by the circumstance that some quantities that appear in classical theory lose their absolute character (the author considers, for instance, three concepts each of which might be called distance). The author does a valiant job, extracting from literature — including his own papers — as many pertinent conclusions as possible; still, the results cannot be considered anything but preliminary.

G. Y. Rainich (Ann Arbor. Mich.).

McVittie, G. C. The non-adiabatic contraction of a gas sphere to a complete polytrope. Astr. J. **61** (1956), 451–462.

In the book reviewed above the author has shown that the class of spherically symmetric motions in a sphere of gas under the action of the pressure gradient and its gravitational self-attraction are described by

$$(1)\qquad q=-\psi_{rt}/\nabla^2\psi,\ \ =\rho-\nabla^2\psi,$$

$$(2)\qquad p=P(t)-\psi_{tt}+2\int\Big(\frac{I}{r}-\frac{4\pi G}{r}\psi_r{}^2\Big)dr+I-2\pi G\psi_r{}^2,$$

where q is the radial velocity, ρ and p are its density and pressure; also a subscript denotes a partial derivative, $I=\psi^2{}_{rt}/\nabla^2\psi$ and the integration is performed as if t were constant and P is an arbitrary function of t. In (1) and (2) $\psi(r, t)$ is undetermined; the author removes this arbitrariness by the supposition

$$(3)\qquad q=f\sigma/f,$$

where f is an arbitrary function of time. On this latter assumption the author investigates the manner in which a sphere of gas initially at rest and not in equilibrium will collapse to a state of final equilibrium. If T denotes the time for collapse and A the initial radius, then in terms of the variables $\zeta=r/f$, $\tau=t/T$ and $R=r/A$, the solution represented by equations (1) and (2) and the assumption (3) becomes

$$q/q_0=R_\tau\zeta;\ \rho/\rho_0=R^{-3}(h_\zeta+2h/\zeta)$$

$$p/p_0=\frac{R_{\tau\tau}}{R^2}H(\zeta)+\frac{\alpha}{R^4}J(\zeta),$$

$$H(\zeta)=\Big[\int h d\zeta+h\zeta\Big]_\xi^1 \text{ and } J(\zeta)=\Big[\int h^2\zeta^{-1}d\zeta+\tfrac{1}{2}h^2\Big]_\xi^1,$$

and q_0, ρ_0 and p_0 are constants and h is an arbitrary function of the argument. The author formulates various conditions which $R(\tau)$ should satisfy and shows that a simple form $R(\tau)$ which satisfies all these conditions is given by $R=1-\frac{1}{3}(1-R_e)(10-15\tau^2+8\tau^3)\tau^2$ where R_e is the final equilibrium radius. To complete the solution one needs the function $h(\zeta)$. This depends not on the assumed form of $R(\tau)$ but on the existance and nature of the final equilibrium. If the latter is assumed to be an Emden polytrope of index n then it is shown that $h=-x_0{}^{-1}dy/dx$ $(x=x_0\zeta)$, where $y(x)$ is the Emden function and x_0 is its first zero. A number of astrophysical questions relating to the formation of stars from intersetllar clouds is discussed in terms of the foregoing solution to the dynamical problem. Particular emphasis is given to the non-adiabatic nature of the collapse and the consequent emission of radiation. *S. Chandrasekhar* (Williams Bay, Wis.).

See also: Functions of Complex Variables: Lense.

Geophysics

Levallois, Jean-Jacques. Sur la détermination du géoïde par des mesures gravimétriques sur la surface topographique. C. R. Acad. Sci. Paris **244** (1957), 2479–2482.

Let a cogeoid, C, be defined as the locus of all points located at the distance $(-h)$ from the topographic surface, T, where h is the height of a point P on T above sea level. Let N_P be the separation between C and a chosen reference ellipsoid E along the vertical through P. By an application of Green's theorem, an intregodifferential formula for N_P is determined which depends on the theoretical gravity computed on E, and the physical gravity measured on T, but not on values of the latter reduced to C, as is usually customary. Methods of solving the integrodifferential formula by approximation are briefly discussed. *B. Chovitz* (Washington, D.C.).

Hansen, Walter. Theorie zur Errechnung des Wasserstandes und der Strömungen in Randmeeren nebst Anwendungen. Tellus **8** (1956), 287–300.

A method is developed for determining the non-stationary currents in estuaries and marginal seas. The equations of motion and continuity neglecting only variation with depth and including the frictional and non-linear terms are formulated in one and two dimensions. These may be replaced by the usual finite difference approximations at points on a line or nodes of a net. With appropriate boundary conditions in both space and time, the problem may thus be solved numerically.

The method has been carried out on BESK digital Computer for tidal currents in the Ems river estuary, and for the heavy storm surge of 1951 in the North Sea. In both these cases, extremely good agreement with observations is obtained.

It is remarked that the use of such a method, combined with numerical weather forecasting, provides a means for the numerical prediction of sea level. *D. C. Gilles.*

Gringorten, Irving I.; Lund, Iver A.; and Miller, Martin A. The construction and use of forecast registers. Air Force Cambridge Research Center, Bedford, Mass., Geophysical Research Papers No. **53** (1956), iv+131 pp.

Les auteurs présentent un „répertoire de prévision" destiné à apporter une aide objective à la prévision d'un certain nombre d'éléments météorologiques. Ce répertoire est essentiellement un relevé aussi détaillé que possible du temps passé traduit en code. Il permet de sélectionner dans le passé les situations „analogues" à la situation météorologique actuelle et d'en déduire l'évolution des phénomènes météorologiques en fonction de celle observée dans ces situations analogues. Les résultats obtenus par l'utilisation du „répertoire de prévision" montrent que la prévision est sensiblement améliorée en général, sauf pour certaines stations. A noter que ce répertoire s'est montré par ailleurs fort utile dans les études climatologiques. *M. Kiveliovitch* (Paris).

Reuter, Heinz. Über die Voraussage zeitlich gemittelter Höhenkarten für Zwecke mittelfristiger Wetterprognosen. Arch. Meteorol. Geophys. Bioklimatol. Ser. A. **9** (1956), 433–438 (1957).

C'est une critique et une mise au point des résultats de J. Namias pour calculer d'avance des cartes de prévision en altitude [Extended forecasting by mean circulation methods U.S. Weather Bureau, Washington, 1947].

L'auteur établit une formule de prévision destinée à calculer une carte moyenne de trois jours servant de base à la prévision à moyenne échéance pour l'Europe.

M. Kiveliovitch (Paris).

Kletter, Leopold. Grenzen der Anwendbarkeit und Erweiterung der Trendmethode von Namias. Arch. Meteorol. Geophys. Bioklimatol. Ser. A. **9** (1956), 439–445 (1957).

L'auteur critique la méthode de Namias (voir l'article précédent) et montre que ses formules ne sont pas toujours valables, sauf dans les cas où la situation isobarique tend vers un état normal. L'auteur décrit un autre procédé capable de dresser des cartes de tendance et qui utilisee la carte synoptique de prévision.

M. Kiveliovitch (Paris).

Spar, Jerome. A note on the stability of baroclinic waves. J. Meteorol. **14** (1957), 136–140.

The stability characteristics of waves in an atmosphere possessing an invariant thermal wind direction are calculated for three different velocity profiles. Values of the critical wavelength, below which waves are unstable, are computed at latitude 40 deg. as functions of the wind shear for each of the three cases. It is found that a jet stream type of velocity profile is the most unstable and that the differences in stability characteristics between the different types of profile increase with increasing wind shear and wavelength. *M. H. Rogers* (Urbana, Ill.).

Elliott, William P. A comparison of some approaches to the diabatic wind profile. Trans. Amer. Geophys. Union **38** (1957), 21–24.

The author compares several expressions for the diabatic mixing length, used in describing the wind profile near the ground when the flow is nonadiabatic.

It is found that the various forms lead to more or less similar profiles: the differences between the profiles being too small for present experimental data to be of much assistance. *M. H. Rogers* (Urbana, Ill.).

See also: Fluid Mechanics, Acoustics: Ellison.

OTHER APPLICATIONS

Economics, Management Science

Berger, Joseph; and Snell, J. L. On the concept of equal exchange. Behavioral Sci. **2** (1957), 111–118.

The paper examines the elementary properties of exchange taking place between every pair of an economic system. Such inter-subsystem exchange may be expressed by a square matrix P whose elements in row give the proportions of each system's initial quantity moving to itself and other systems so that they sum up to unity: $P=(P_{ij})$ and $\sum_j P_{ij}=1$. The definitions of concepts necessary to analyze 'equilibrium', 'equal exchange', 'strict exchange' of the system and some pertinent theorems are given. Among those the following theorem seems important: Let P have all nondiagonal entries positive. A necessary and sufficient condition for such a P to define a strict exchange system is that for some fixed r and every pair k, j

$$p_{rj}p_{jk}p_{kr}=p_{rk}p_{kj}p_{jr}.$$

Here strict exchange is defined by P if every pair of the system sends out and receives an equal quantity. Applications to problems of social mobility are given. References are made to the theory of Markov chains and S. J. Prais's papers [J. Roy. Statist. Soc. Ser. A. **118** (1955), 56–66; Population Studies **9** (1955), 72–81].

S. Ichimura (Osaka).

Pfouts, R. W. Distribution theory in a certain case of oligopoly and oligopsony. Metroecon. **7** (1955), 137–146.

A concise summary of previously established results.

S. Ichimura (Osaka).

Stuvel, G. A new index number formula. Econometrica **25** (1957), 123–131.

The author has "stumbled" on a new index formula:

$$P_n=X_n+Y_n, \quad Q_n=-X_n+Y_n,$$

where $X_n=\frac{1}{2}(\Lambda_n-L_n)$, $Y_n=(X_n^2+V_nV_0^{-1})^{\frac{1}{2}}$,

$V_n=\Sigma p_n q_n$, $V_0=\Sigma p_0 q_0$, $\Lambda_n V_0=\Sigma p_n q_0$, $L_n V_0=\Sigma p_0 q_n$. Λ_n denotes thus the price index, L_n the volume index of Laspeyres, and V_n the value of the varying commodity basket at point of time n.

The formulas are interesting because they fulfil many of the tests for index formulas formulated by I. Fisher in his book "The making of index numbers" [Houghton-Mifflin, Boston-New York, 1922]. They were found by splitting the increase V_n-V_0 in two parts A_n+B_n, where

$$A_n=\Sigma\left(\frac{p_n+p_0}{2}\right)(q_n-q_0), \quad B_n=\Sigma\left(\frac{q_n+q_0}{2}\right)(p_n-p_0),$$

and putting

$$A_n=\left(\frac{P_n+1}{2}\right)(Q_n V_0-V_0), \quad B_n=\left(\frac{Q_n+1}{2}\right)(P_n V_0-V_0).$$

The author uses (p, q) instead of (P, Q), and vice versa, and not the usual notation used by me. I must say that I did not myself stumble on these index formulas even though I have already for about twenty years made use of the same split

$$V_n-V_0=A_n+B_n$$

for the increase in the value of an inventory in a part A_n due to changes in quantities and a part B_n due to changes in prices. The author has thus made an interesting and not quite trivial discovery in the theory of index numbers.

It might be of interest to notice that this index has the property:

$$\Lambda_n-L_n=P_n-Q_n, \quad P_n Q_n=V_n/V_0,$$

while Fisher's ideal index has the property:

$$\Lambda_n/L_n=P_n(F)/Q_n(F), \quad P_n(F)Q_n(F)=V_n/V_0.$$

It is thus to be expected that Stuvel's index will in practice be almost equal to Fisher's ideal index. Their difference can in fact be written in the form

$$P_n-P_n(F)=\frac{(\Lambda_n-L_n)(\Lambda_n-P_n(F))}{(P_n+L_n)-(\Lambda_n-P_n(F))}.$$

If, for instance, $\Lambda_n-P_n(F)=0.1$, $\Lambda_n-L_n=0.1$, $P_n+L_n=2$, we will get

$$P_n-P_n(F)=0.01/1.9=0.0053.$$

A difference of the order of size of 10% between the price index of Laspeyres and Fisher occures only in rare practical cases. Stuvel's index will thus only rarely differ from Fisher's ideal index by an amount large enough to be seen in the figures used for presenting the results.

L. Törnqvist (Helsinki).

Castiglioni, Pietro. Considerations on the construction of a mathematical theory of money. Metroecon. **7** (1955), 85–94.

The author gives a systematized account of different attempts to form a theory of money value. He contends that the only theory, which gives due consideration to all the functions of money and is at the same time consistent with static as well as dynamic approaches, is one where the utility of the services of money, i.e., the amount of money divided by a price index, is included in the economic subjects' utility function. *S. Malmquist.*

Klein, L. R. On the interpretation of Theil's method of estimating economic relationships. Metroecon. **7** (1955), 147–153.

It is shown that Theil's "two-rounds estimates" method [Econometrica **23** (1955), 204–205; and "Estimation and simultaneous correlation in complete equation systems", Central Plan-Bureau, The Hague, 1953] may be interpreted in terms of the well-known method of instrumental variables [cf. R. C. Geary, Econometrica **17** (1949), 30–58; MR **10**, 465, 856], and that it is related to the limited information method in a certain way. In this connection the following theorem is proved: Two-rounds estimates of the parameters in equation

$$x_t=\sum_{i=1}^{n}\alpha_i y_{it}+\sum_{i=1}^{r}\beta_i z_{it}+u_t,$$

are the same as estimates obtained by solving

$$\begin{pmatrix} M_{\hat{y}x} \\ M_{z^*x} \end{pmatrix}=\begin{pmatrix} M_{\hat{y}y} & M_{\hat{y}z^*} \\ M_{z^*y} & M_{z^*z^*} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix},$$

where $\hat{y}_{it}$ are the values of y_{it} calculated from the least-squares regression of y_{it} on all the elements of z_t, and M's are the moment matrices among elements of $\hat{y}$ and x and z.

S. Ichimura (Osaka).

★ **Goode, Harry H.; and Machol, Robert E. System engineering: An introduction to the design of large-scale systems.** McGraw-Hill Book Company, Inc., New York-Toronto-London, 1957. xii+551 pp. $10.00.

The exact dividing line between systems engineering and operations research is not sharp. The former tends to be more interested in "gadgets" and the latter in the users of the gadgets. Many of the problems of operation research have to do with finding ways of using most effectively the equipment one has, whereas most of the problems of systems engineering relate to the design of new equipment which, it is hoped, will enable the user to do a better job. Though each must have some of the other's skills, they do have different jobs to do.

The present book does not make particularly clear just what is or is not to be included in the field of systems engineering. The subjects discussed range from the elements of probability, via the components and operation of digital and analogue computers, to game theory, to three pages of linear programming, a couple of dozen on communication theory and a small amount of queuing theory, and thence to an "epilogue" on economics. A number of examples, presumably of systems engineering problems, are quoted from texts and articles on operations research. On the other hand very little is said about equipment design itself and but twenty pages are devoted to servo theory. Perhaps the book is to be considered as an introduction to the ancillary subjects the systems engineer should know something about, after he has obtained his basic engineering training.

Problems are given at the end of several chapters. There is a useful and fairly representative bibliography.

P. M. Morse (Cambridge, Mass.).

Del Vecchio, Ettore. Sul calcolo approssimato di premi di assicurazioni sulla vita al variare del tasso d'interesse. Giorn. Mat. Finanz. (4) **1** (1955), 111–129.

Begründung der Methode von E. Lenzi, welcher die Zinsinterpolation bei Leibrenten auf diejenige von Zeitrenten zurückführte. Anwendung der entsprechenden Näherungsformeln auf die Prämien lebenslänglicher Todesfallversicherungen und gemischter Versicherungen.

W. Saxer (Zürich).

Bonferroni, Carlo. I valori mediani e la teoria della misura. Giorn. Mat. Finanz. (4) **1** (1955), 89–110.

Vortrag des Verf. über die verschiedenen Mittelwerte

und Mediane von statistischen endlichen und unendlichen Folgen sowie Funktionen. Definition symmetrischer Funktionale und ihr Zusammenhang mit den Mittelwerten.
W. Saxer (Zürich).

de Jongh, B. H. Ein System von gegenseitig stochastisch abhängigen Risiken. Verzekerings-Arch. Actuarieel Bijvoegsel **34** (1957), 66*–72*.

Betrachtung eines mathematischen Modelles mit mehreren Risiken, bei denen Schäden verschiedener generationen unterschieden werden. *W. Saxer* (Zürich).

Campagne, C.; und Driebergen, C. Das Solvabilitätskriterium in der Schadenversicherung. Verzekerings-Arch. Actuarieel Bijvoegsel **34** (1957), 43*–65*.

Exakte und approximative Berechnung der Rinnwahrscheinlichkeit im Falle allgemeiner Risikoversicherungen und praktische Folgerungen. *W. Saxer* (Zürich).

Leimbacher, W. Zur Umwandlung temporärer Extraprämien in äquivalente Summendreduktionen. Mitt. Verein. Schweiz. Versich.-Math. **57** (1957), 37–43.

Sub-standard risks are usually insured under the condition that an extra premium is collected for a certain period. Alternatively, one could reduce the risk sum during the same period. The author derives approximation formulae which express the reduction of risk sum in terms of the equivalent extra premium. *E. Lukacs.*

Amsler, M. H. Zur maschinellen Auflösung des Zinsfussproblems. Mitt. Verein. Schweiz. Versich.-Math. **56** (1956), 259–273.

See also: Computing Machines: Ting Kuan. **Biology and Sociology:** Beckmann.

Programming, Resource Allocation, Games

★ **Luce, R. Duncan; and Raiffa, Howard. Games and decisions: introduction and critical survey.** A study of the Behavioral Models Project, Bureau of Applied Social Research, Columbia University; John Wiley & Sons, Inc., New York, N. Y., 1957. xix+509 pp. $8.75.

Since 1944 when "The theory of games and economic behavior" [Princeton; MR **6**, 235] by John von Neumann and Oskar Morgenstern appeared and inspired a continuing spate of research, many workers in the vineyard have come to refer affectionately to this pioneering work as the "bible". Consequently there is some temptation to dub the work under review the "new testament" or perhaps the "gospel according to Luce and Raiffa", but these terms would be somewhat misleading. "Games and Decisions" is an introduction to and a book of commentaries on the "bible", as well as a summary of the more recent revelations of the research of the bakers' dozen years which clapsed between the two publication dates.

The authors' preface asserts that "this book attempts to communicate the central ideas and results of game theory and related decision-making models unencumbered by their technical mathematical details; thus, for example, almost no proofs are included. It is a book about game theory, not a presentation of the theory itself. By laying bare the main structure of the theory — its assumptions and conclusions, its deficiencies and aspirations — we hope that the book will serve as a useful critical introduction to the theory and a guide to the literature. We have tried, on the one hand, to make sufficiently precise statements so that misunderstandings and misstatements will not result from a reading of the book, but, on the other hand, we have striven to keep the language and notation sufficiently familiar and simple that there will be scientists who will benefit from it who would have found a treatise on game theory unintelligible."

The authors accomplish this perilous journey over eggshells with good judgment and success. Perhaps the book's strongest feature is the deftness with which they lead the reader through the heuristic considerations which precede the precise mathematical formulations. While not every reader will agree with every opinion of the authors', the conceptual criticism, which is the book's main preoccupation, is always thought-provoking and clearly expressed.

Linear programming is discussed briefly as an example of decision making under certainty, and its connections with game theory are explained. Utility theory is discussed at length as decision making under risk. The bulk of the book is devoted to game theory proper. Extensive and normal forms, and the minimax theorem for two-person zero-sum games are discussed, as well as two-person non-zero-sum games both cooperative and non-cooperative. Thus, Nash's bargaining problem, the Shapley value, and Raiffa's arbitration schemes are discussed and related to each other. The theory of n-person games is then taken up at length. The von Neumann-Morgenstern theory of characteristic functions and solutions is treated, as well as the many modifications and alternatives which have been proposed, such as Nash's equilibrium points, Luce's φ-stability, Milnor's reasonable outcomes, and Shapley's value. Statistical decision making is discussed in its relation to games. The final chapter discusses in detail the Arrow social choice paradox. Many more technical or more peripheral (from the social scientist's point of view) recent contributions are summarized briefly. The book is also interlarded with suggestions as to the possible future development of the theory.

One hundred and fourteen pages of appendices take the reader a little further into the mathematical details of many subjects, still keeping the mathematical level quite elementary. While these will not satisfy the mathematically inclined reader, they will serve to introduce him gently to and stimulate him to read further in the more strenuous periodical and expository literature cited. An extensive bibliography lists published material up to date, even including some items not yet in print.

The book should be welcomed by all interested in acquiring some knowledge of the potentialities and limitations of mathematical models in the social sciences without becoming embroiled in the mathematical details. It should be indispensable supplementary reading for students in courses in the mathematical theory of games and linear programming which tend naturally to concentrate on the technical mathematical difficulties. And it should serve also to remind mathematical readers that the ultimate usefulness of applied mathematics still depends, as always, on success in negotiating the treacherous two-way bridge of abstraction and interpretation which links reality with the mathematical model.

M. Richardson.

★**Duffin, R. J. Infinite programs.** Linear inequalities and related systems, pp. 157–170. Annals of Mathematics Studies, no. 38. Princeton University Press, Princeton, N. J., 1956. $5.00.

In a locally convex real linear space U, let $u \geqq 0$ mean $u \in U_+$, where U_+ is a given convex cone; similarly for another space V and cone V_+, and for the conjugate spaces U^* and V^* and polar cones U_+^* and V_+^*. A "program" consists of a given $a \in U$, $b \in V$, and a linear transformation A from U^* into V. Its "value" M and "subvalue" $m \leq M$ are $\inf(a, x)$, $x \geqq 0$, $Ax \geqq b$, $\inf \lim \inf(a, x_n)$, $x_n \geqq 0$, $Ax_n \geqq b_n$, $b_n \to b$. Its "dual", with value M' and subvalue m', consists of $-b$, $-a$, $-B$, where B is the "adjoint" of A, satisfying $(Au^*, v^*) = (Bv^*, u^*)$. Typical results: M is finite if and only if m' is finite; $M + m' = 0$; M' is attained if and only if (i) M is finite, (ii) $x \geqq 0$, $Ax > b$ is solvable; in this case $M = m$. Application to special selfdual programs. *T. S. Motzkin.*

Fan, Ky; Glicksberg, Irving; and Hoffman, A. J. Systems of inequalities involving convex functions. Proc. Amer. Math. Soc. **8** (1957), 617–622.

Let $f_i(x)$ $(1 \leq i \leq m)$ be m convex functions defined on a nonempty convex set K in a real vector space. Then one and only one of the following two alternative holds: either the system $f_i(x) < 0$ $(1 \leq i \leq m)$ is consistent, i.e. has a solution in K, or there exist numbers λ_i, nonnegative and not all zero, such that $\sum_{i=1}^{m} \lambda_i f(x) \geqq 0$ in K. This result and a number of applications, all proved very simply, contain several earlier theorems as special cases.
W. K. Hayman (London).

Peck, J. E. L.; and Dulmage, A. L. Games on a compact set. Canad. J. Math. **9** (1957), 450–458.

The main result of this paper is a general minimax theorem which, as the authors note, was previously published by B. J. Pettis [Math. Mag. **29** (1956), 233–247; MR **17**, 1111]. The authors develop some important consequences and special cases, and an extension involving mixed strategies and payoff functions vanishing at infinity on a locally compact space of pure strategies. Pettis' theorem is shown to contain several previous minimax theorems, and is compared with still others in two pages of examples. *J. Isbell* (Seattle, Wash.).

Castañeda, José. Introduction to game theory and its applications. I. Two-person zero-sum games. Rev. Ci. Apl. **11** (1957), 104–118. (Spanish)

Castañeda, José. Introduction to game theory and its applications. II. n-person games and games with non-zero sum. Application to a duopoly. Rev. Ci. Apl. **11** (1957), 204–220. (Spanish)

This is a brief survey of expository character of the Theory of Games. The proofs of the most involved theorems are omitted. There is a number of well chosen examples and the application to a practical case of duopoly is presented in some detail. *A. G. Azpeita.*

See also: Foundations, Theory of Sets, Logic: Richardson. **Linear Algebra:** Guttman.

Biology and Sociology

Kerner, Edward H. A statistical mechanics of interacting biological species. Bull. Math. Biophys. **19** (1957), 121–146.

Urbanik, K. Remarks on the maximum number of bacteria in a population. Zastos. Mat. **2** (1956), 341–348. (Polish. Russian and English summaries)

Consider a birth and death process with η and $\gamma = a\eta$ denoting the constant intensities of births and deaths. Let $\xi(t)$ denote the number of individuals living at time t and $S = \max \xi(t)$ over $0 \leq t \leq \infty$. The first result obtained is that $P\{S < k | \xi(0) = r\} = (a^r - a^k)/(1 - a^k)$ for $a \neq 1$ and $= (k - r)/k$ if $a = 1$. The second result is concerned with conditional expectation, say τ, of the time when $\xi(t)$ attains, for the first time, its absolute maximum S, given that this maximum is finite. It is found that for τ to be finite it is necessary and sufficient that $\gamma \neq \eta$. *J. Neyman.*

Beckmann, Martin J. On the equilibrium distribution of population in space. Bull. Math. Biophys. **19** (1957), 81–90.

This paper sets up a model, from which one can in principle derive equilibrium distributions of populations in an economic region. The author assumes that the net immigration into the region is zero, and accordingly is concerned only with internal migration. His model, which is hydrodynamic in nature, states in mathematical language that an equilibrium distribution of population requires a balance in the migration flows induced by differences in income levels, population densities, and reproduction rates over the region studied. If one specializes to the case where essentially all locations in a region are equally desirable, the model predicts a steady state distribution of population which is uniform over the region. If one considers the special case where the annual population surplus at a point depends linearly on the population density, then it can be shown that one can describe conditions of equilibrium by means of a "potential of population", a concept introduced into demography by J. Q. Stewart. *B. Epstein* (Stanford, Calif.).

Rashevsky, N. Contributions to the theory of imitative behavior. Bull. Math. Biophys. **19** (1957), 91–119.

See also: Statistics: Bliss. **Economics, Management Science:** Berger and Snell.

Information and Communication Theory

★**Siforov, V. I. Parameters of binary coding systems.** Translated by Morris D. Friedman, 572 California St., Newtonville 60, Mass., 1956. 13 pp.

The systems discussed are binary transmission systems with zero memory and constant probability that any symbol is incorrectly transmitted. Definitions are introduced of the probable decoding error per symbol, the relative probable decoding error, the quality of a coding system, and the channel use coefficient. The physical meaning of these parameters is shown. *J. Wolfowitz.*

Brookes, B. C. An introduction to the mathematical theory of information. Math. Gaz. **40** (1956), 170–180.

Jacobsen, B. B. Probability theory in telephone transmission. Teleteknik **1** (1957), 83–85.

See also: Topological Vector Spaces: Gel'fand. **Statistics:** Lindley. **Statistical Thermodynamics and Mechanics:** Jaynes.

Control Systems

Higgins, T. J. A résumé of the development and literature of nonlinear control-system theory. Trans. A.S.M.E. **79** (1957), 445–449, discussion 449–453.

This paper is a valuable summary of the classical and modern theory of nonlinear differential equations, and its interrelation with electronic and mechanical circuit theory.

R. Bellman (Santa Monica, Calif.).

Stout, T. M. Basic methods for nonlinear control-system analysis. Trans. A.S.M.E. **79** (1957), 497–507, discussion 507–508.

The methods of phase analysis and harmonic analysis, common in the study of non-linear oscillations, are described in the terminology of control-systems and are applied to some simple non-linear control-systems. Brief mention is made of possible applications of numerical procedures. *H. B. Keller* (New York, N.Y.).

Popov, E. P. On the use of the harmonic linearization method in the automatic control theory. NACA Tech. Memo. no. **1406** (1957), 6 pp.

Translated from Dokl. Akad. Nauk SSSR (N.S.) **106** (1956), 211–214; MR **17**, 851.

★**Povarov, G. N. To the mathematical theory of the synthesis of (1, *k*)-terminal switching networks.** Translated by Morris D. Friedman, 572 California St., Newtonville 60, Mass., 1956. 5 pp.

Translation of Dokl. Akad. Nauk SSSR (N.S.) **111** (1956), 102–104.

★**Lunts, A. G. Method of synthesis of (1, *k*) terminal networks.** Translated by Morris D. Friedman, 572 California St., Newtonville 60, Mass., 1956. 5 pp.

Translated from Dokl. Akad. Nauk SSSR (N.S.) **112** (1957), 55–57.

Constantinescu, Paul. Sur la réduction du nombre de contacts en introduisant les circuits en pont. Conductibilités directes conjuguées. Acad. R. P. Romîne. Stud. Cerc. Mat. **7** (1956), 399–419. (Romanian. Russian and French summaries)

Moisil, Gr. C. La simplification des schémas par introduction de contacts à plusieurs positions. Acad. R. P. Romîne. Bul. Ştiinţ. Secţ. Ştiinţ. Mat. Fiz. **7** (1955), 843–852. (Roumanian. Russian and French summaries)

L'Auteur étudie la simplification des schémas lorsqu'on introduit des contacts à plusieurs positions. Un pareil contact peut être représenté de deux manières différentes, comme des ensembles de contacts à deux positions (représentation *A* et *S*).

On déduit deux méthodes de simplification.

Résumé de l'auteur.

Moisil, Gr. C. Une méthode de synthèse pour les schémas à contacts et relais. Acad. R. P. Romîne. Fil. Iaşi. Stud. Cerc. Ştiinţ. **6** (1955), 19–27. (Roumanian. Russian and French summaries)

Dans ce travail, l'auteur propose une méthode de synthèse des schémas à contacts et relais lorsqu'on donne les programmes de travail des éléments executifs pour des programmes donnés pour les éléments réceptifs.

Résumé de l'auteur.

Moisil, Gr. C.; et Nedelcu, Mariana. Analyse, synthèse et simplification des schémas à contacts et soupapes, avec commande directe. Acad. R. P. Romîne. Bul. Ştiinţ. Secţ. Ştiinţ. Mat. Fiz. **8** (1956), 469–487. (Romanian. Russian and French summaries)

Le présent travail étudie l'analyse, la synthèse et la simplification des schémas Π contenant des redresseurs. On attache à un redresseur la valeur ε ou $\bar{\varepsilon}$, suivant qu'il laisse le courant traverser le schéma dans un sens ou dans l'autre.

On étudie deux problèmes d'analyse: dans le premier, on donne la situation des contacts et on demande la détermination du schéma; dans le second, on demande la position des contacts pour une certaine conductibilité du schéma.

Dans ce dernier cas, la formule de structure devient:

$$\varphi = g + h\varepsilon + k\bar{\varepsilon}$$

avec $gh = gk = hk = 0$ et elle permet la synthèse du schéma lorsque les conditions pour les fonctions g, h, k sont données.

Le travail contient quelques examples de calcul.

Résumé de l'auteur.

Ioanin, Gh. Synthèse des schémas où figurent des sélecteurs. Acad. R. P. Romîne. Bul. Ştiinţ. Secţ. Ştiinţ. Mat. Fiz. **8** (1956), 489–500. (Romanian. Russian and French summaries)

On décrit la construction et le principe de fonctionnement des sélecteurs pas à pas et l'on montre que leur fonctionnement peut être décrit algébriquement à l'aide des fonctions établies ou des contacts à positions multiples. En guise d'application, on fait la synthèse de quelques schémas où figurent des sélecteurs pas à pas.

Résumé de l'auteur.

Moisil, Gr. C. Théorie algébrique du fonctionnement en plusieurs temps des schémas à contacts et relais à deux éléments intermédiaires. Acad. Repub. Pop. Romîne Fil. Cluj. Stud. Cerc. Ştiinţ. **5** (1954), 7–14. (Romanian. Russian and French summaries)

Moisil, Gr. C. Synthèse des schémas à relais idéaux, à l'aide des corps d'imaginaires de Galois. Acad. R. P. Romîne. Bul. Ştiinţ. Secţ. Ştiinţ. Mat. Fiz. **8** (1956), 429–453. (Romanian. Russian and French summaries)

Si ξ^t décrivent l'état des n relais d'un schéma ($\xi^t=1$: le courant passe par le bobinage du relais X_t; $\xi^t=0$: le courant n'y passe pas), l'auteur introduit le nombre $\xi=\sum \xi^t \varepsilon^{t-1}$, où ε est un élément primitif du corps de Galois GF(2^n); la valeur de x pendant le N-ième période temporel est ξ_N. Pour les n contacts des relais et pour les m boutons de commande on a les nombres analogues x_N et α_N. Alors $\xi_N = F(\alpha_N, x_N)$; l'équation de recurrence du schéma est $x_{N+1} = F(\alpha_N, x_N)$. Application aux cas $n=2, 3, 4$.

O. Bottema (Delft).

Moisil, Gr. C.; et Popovici, Constantin P. Analyse et synthèse des schémas à commande directe, à l'aide des imaginaires de Galois. Acad. R. P. Romîne. Bul. Ştiinţ. Secţ. Ştiinţ. Mat. Fiz. **8** (1956), 455–467. (Romanian. Russian and French summaries)

Un schéma à commande directe est formé uniquement d'éléments exécutifs et de boutons de commande. On introduit au système des boutons l'élément α de GF(2^m) (cf. l'article précédent) et à chaque élément exécutief W une

variable w de GF(2). Alors $w=F(\alpha)$, F étant la fonction de travail de W, qui s'obtient à l'aide de la formule d'interpolation de Lagrange. Discussion des racines de $F(\alpha)=0$ et $F(\alpha)=1$. *O. Bottema* (Delft).

Moisil, Gr. C. L'emploi des imaginaires de Galois dans la théorie des mécanismes automatiques. II. Schémas à deux éléments intermédiaires. Com. Acad. R. P. Romîne **4** (1954), 587–589. (Romanian. Russian and French summaries)

For parts I, III see MR **17**, 1180; **18**, 784.

Moisil, Gr. C. Utilisation des imaginaires de Galois dans la théorie des mécanismes automatiques. IV. Une théorie trivalente des relais polarisés. Com. Acad. R. P. Romîne **6** (1956), 505–508. (Romanian. Russian and French summaries)

A network is considered, involving bi-polar, three-way switches, connected arbitrarily in series or in parallel. Each such network can be represented by a formula which (in canonical form) reads: (*) $\sum \lambda_{\alpha_1 \cdots \alpha_n} K_1^{\alpha_1} \cdots K_n^{\alpha_n}$. Here $\alpha_j=0, 1, 2$ ($j=1, 2, \cdots, n$) denotes one of the three possible positions of the switch K_j and $\lambda_{\alpha_1 \cdots \alpha_n}=0$, or 1. Let $F(K_1, \cdots, K_n)$ be a function on the set of switches such that $F=1$ if current can flow through the network, $F=0$ otherwise. By use of Lagrange interpolation polynomials in GF(3), it is shown that $F(K_1, \cdots, K_n)$ can be obtained algebraically from (*). Application is made to a three-way switch, acted upon by a polarised relay. In the case of n relays, one uses GF(3^n) instead of GF(3). *E. Grosswald.*

Moisil, Gr. C. Utilisation des imaginaires de Galois dans la théorie des mécanismes automatiques. V. Classification des évolutions des schémas avec un relai. Com. Acad. R. P. Romîne **6** (1956), 509–513. (Romanian. Russian and French summaries)

With the conventions made in previous parts of this sequence, the evolution of a network with an ordinary relay is governed by a recurrence relation (1) $x_{N+1}=ax_N+b$ with $a, b, x_N \in$ GF(2). It is shown that (1) can be reduced to three canonical forms: $x_{N+1}=1$, $x_{N+1}=x_N$, $x_{N+1}=x_N+1$. If the relay is polarized, (2) $x_{N+1}=ax_N^2+bx_N+c$ with $a, b, c, x_N \in$ GF(3) and it is shown that (2) may be reduced to one of seven canonical forms. *E. Grosswald* (Philadelphia, Pa.).

Vaida, Dragoş. Application des imaginaires de Galois à la théorie des mécanismes automatiques. VI. Classification des évolutions des schémas à deux éléments intermédiaires. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **8** (1956), 21–29. (Romanian. Russian and French summaries)

Moisil has shown [Com. Acad. R. P. Romîne **4** (1954), 581–585, MR **17**, 1180; see also the paper listed third above] that the successive states of relay-systems without time-lag, involving two intermediate elements, are governed by the recurrence relation (*) $Z_{N+1}=aZ_N^3+bZ_N^2+cZ_N+d$, with constant coefficients in GF(2^2). However, not all these relations lead to essentially distinct succession schemes. Using further results by Moisil [see the paper reviewed second above], the author finds necessary and sufficient conditions for two recurrence relations to be isomorphic, i.e., to lead to essentially the same succession of states; it then follows that all possible 256 relations (*) fall into 19 essentially distinct classes, each represented by a canonical form. *E. Grosswald* (Philadelphia, Pa.).

Moisil, Gr. C. L'emploi des imaginaires de Galois dans la théorie des mécanismes automatiques. VII. Relais polarisés réels. Com. Acad. R. P. Romîne **6** (1956), 621–623. (Romanian)

A "real" three-way switch moves continuously from one contact to another; hence there are also two intermediate (no contact) positions. Taking these into account, a three-way switch has 5 possible states. The theory of Part IV [see the paper reviewed third above] may now be repeated, using Lagrange interpolation in GF(5) instead of in GF(3). *E. Grosswald* (Philadelphia, Pa.).

Moisil, Gr. C. L'emploi des imaginaires de Galois dans la théorie des mécanismes automatiques. VIII. Relais basculants réels. Com. Acad. R. P. Romîne **6** (1956), 625–626. (Romanian)

The theory of previous parts is applied to relays acting upon switches with four possible states (two neutral). Here use is made of GF(2^2). *E. Grosswald.*

Moisil, Gr. C. L'emploi des imaginaries de Galois dans la théorie des mécanismes automatiques. IX. Classification des schémas avec un bouton et un relais. Com. Acad. R. P. Romîne 6 (1956), 1055–1058. (Romanian. Russian and French summaries)

The evolution of a network, containing a simple relay and a push-button, is considered. The 16 possible recurrence relations $x_{N+1}=\alpha a x_N+\beta a+\gamma x_N+\delta$ (all variables in GF(2), $\alpha, \beta, \gamma, \delta$ arbitrary) are shown to reduce to seven canonical types. *E. Grosswald.*

Moisil, Gr. C. Utilisation des logiques trivalentes dans la théorie des mécanismes automatiques. III. Schémas à contacts réels. Com. Acad. R. P. Romîne **6** (1956), 385–386. (Romanian. Russian and French summaries)

Part II is listed in MR **18**, 712.

Moisil, Gr. C. L'emploi des logiques trivalentes dans la théorie des mécanismes automatiques. IV. Réalisation des fonctions de travail au cours du fonctionnement réel. Com. Acad. R. P. Romîne **6** (1956), 971–973. (Romanian. Russian and French summaries)

See also: Ordinary Differential Equations: Shen. **Mechanics of Particles and Systems:** Fogagnolo Massaglia.

AUTHOR INDEX

(Continued from cover 2)

Couty, R. 316
Covey, W. See Halstead, M. H.
Cowley, J. M.-Moodie, A. F. ... 357
Coxeter, H. S. M. 248, 304
Coz, M. 306
Crausse, É.-Poirier, Y. 354
Craya, A. 351
Cristescu, R. 293
Crum, M. M. 277
Császár, Á. 326
Csukás, A. See Ladik, J.
Cunliffe, A. See Gould, R. N.
Curtet, R. 348
Czerwiński, Z. 237
*Czwalina, A. 347
Dalla Volta, V. 314
Danilyuk, I. I. 260, 282
Das, M. N. 331, 332
Datsev, A. B. 360
Davis, P. 268
Dedecker, P. 302
De Dominicis, C. See Brown, G. E.
Delsarte, J.-Lions, J. L. 273
Del Vecchio, E. 372
Deprit, A. 241, 293
Deresiewicz, H.-Mindlin, R. D. . 341
Descombes, R. 253
De Socio, M. 358, 359
Dhen, W. 325
Dickinson, D. 263
Diederich, F. W. See Eggleston, J. M.
Dieudonné, J. 290
Diliberto, S. P. 275
DiPrima, R. C. See Carrier, G. F.
Dittmann, G. 250
Divinsky, N. 245
Długač, M. I. 336
Dolaptschiew, B.-Tschobanow, I. 273
Dolbeault, P. 316
Dombrovskiĭ, G. A. 348
*Dorfner, K.-R. 352
Dowker, C. H. 301
Drazin, M. P. 246
Dressel, F. G. See McLeod, R. M.
Driebergen, C. See Campagne, C.
Drîmbă, Constanța.-Drîmbă, Constantin. 306
Drîmbă, Constantin. See Drîmbă, Constanța.
Duffin, R. J. 261, 374
Dulmage, A. L. See Peck, J. E. L.
Dumitrescu, D. 347
Dwass, M. 331
Dwinger, P. 240, 243
Džrbašyan, M. M. 255
Edwards, D. A. 295
Effertz, F. H. 360
Eggleston, J. M.-Diederich, F. W. 350
Egorov, V. 300
Ehlers, J. 367
Elcock, E. W.-Landsberg, P. T. 335
Elliott, J. P.-Judd, B. R.-Runciman, W. A. 346
Elliott, W. P. 371
Ellis, J. W. 290
Ellison, T. H. 351
Englert, F. 362
Englert, G. W. 349
Enz, C. P. 364
Erdös, P. 252
Ericksen, J. L. 283
Eriksson, F. 308
Eringen, A. C. 340
Eshelby, J. D. 338
Evangelisti, G. 288
Evan-Iwanowski, R. M. 339
Ewing, G. M. See Brunk, H. D.
Fadle, J. 274
Fage, M. K. 273
Faln, V. M. 335
Fan, Chung-Teh. 333
Fan, Chung-Teh. See Swineford, F.
Fan, Ky.-Glicksberg, I.-Hoffman, A. J. 374
Farahat, H. K. 242
Fava, F. 282
Faymon, K. See Kuerti, G.
Fekete, M.-Walsh, J. L. 263
Fell, J. M. G. 294
Feller, W.-McKean, H. P., Jr. 327
Fialkow, A. D. 360
Fichera, E. 334
Fiedler, M. 303
Finzi, L. 342
Fitch, F. B. 237
Flett, T. M. 266, 305
Floyd, E. E. 292, 301
Fogagnolo Massaglia, B. .. 276, 334
Frahn, W. E.-Lemmer, R. H. .. 365
Frank, E. 253
Franz, W. See Beckmann, P.
Fréchet, M. 307, 325
Frederiksen, E. 354
Freudenthal, H. 289
Friedl, W. See Hartenstein, B.
Furry, W. H. 334
Galafassi, V. E. 318
Galimov, K. Z. 336, 339
Garabedian, P. R. 348, 349
García, C. I. 263
Gaudfernau, L. 325
Gavrilov, N. I. 275
Gavrilyuk, V. T. 255
Gel'fand, I. M. 293
Gell-Mann, M. 366
Gerard, G. 343
Gergen, J. J. See McLeod, R. M.
Germer, H. 321
Gerstenkorn, H. 368
Gilvarry, J. J. 365
Girault, M. 326
Giuliano, L. 273
Gladkiĭ, A. V. 255
Glaser, V.-Jakšić, B. 364
Glicksberg, I. See Fan, Ky.
Gnedenko, B. 331
Godeaux, L. 318
Godfrey, D. E. R. 337
*Godnev, I. N. 336
Godunov, S. K. 353
Goetz, A. 299
Gołab, S. 312
Golden, S. 364
Golecki, J. 338
Gomboli, G. 321
Gomes, R. L. 367
*Goode, H. H.-Machol, R. E. .. 372
Goodman, A. W. 260
Goodman, L. E. 340
Gorowara, K. K. 306
Górski, J. 262
Gould, R. N.-Cunliffe, A. 359
*Gould, S. H. 287
*Grammel, R. 336
Grard, F. 365
Grauert, H.-Remmert, R. 317
Green, H. F. 264
Greenberg, B. G. See Sarhan, A. E.
Greville, T. N. E. 243
Griesel, H. 314
Grinberg, G. A.-Lebedev, N. N.-Skalskaia, I. P.-Uflyand, Ya.S. 359
Gringorten, I. I.-Lund, I. A.-Miller, M. A. 371
Groschwitz, E.-Siebertz, K. 345
Grothendieck, A. 315
Gruenbaum, B. 304
Grümm, H.-Spurny, H. 358
Grün, F. See Batschelet, E.
Grzegorczyk, A. 238
Gugenheim, V. K. A. M.-Spencer D. C. 310
Gulmanelli, P.-Montaldi, E. ... 364
Gulotta, B. 325
Gumowski, I. 274
Gundlach, K.-B. 260
Gunning, R. C. 315
Gupta, Hansraj. 252
Guttman, L. 242
*Haack, W. 305
Handane, J. B. S. 330
Hale, J. K. 276
Halfin, L. A. 362
Halilov, Z. I. 284, 286
Halim, E. 260
Halstead, M. H.-Richman, R. L.-Covey, W.-Merryman, J.D. 325
Hamaguchi, M. 363
Hamilton, O. H. 301
Hammer, P. C.-Wymore, A. W. 323
Hanai, S. See Morita, K.
Hannan, E. J. 333
Hansen, W. 371
Hanus, W.-Rayski, J. 366
Harazov, D. F. 295
Harish-Chandra. 292, 293
Harris, T. E. 238
Harrold, O. G., Jr. 300
Hartenstein, B.-Friedl, W. 359
Hartman, P. 296
Hawkins, D.-Mientka, W. E. ... 241
Heilman, O. 340
Heppes, A. 304
Hewitt, E.-Wigner, E. P. 270
Hewitt, E.-Zuckerman, H. S. .. 249
Hida, K. 353
Hieke, M. 343
Higgins, T. J. 375
Hille, E. 277, 327
Hiramatu, H. 312
Hirzebruch, F. 317
Hoelzer, H. See Seebeck, C. L., Jr.
Hoffman, A. J. See Fan, Ky.
Holladay, J. C. 293
Holmann, H. 317
Hönl, H.-Maue, A.-W. 367
Hopf, E. 350
Hopkins, H. G. 342
Horváth, J. 270
Horvay, G.-Born, J. S. 337
Hua, Loo-Keng. 260
Hubbard, J. 362
Huber, A. 345
* van de Hulst, H. C. 359
Hummel, J. A. 267
Iacob, C. 349
Ibragimov, I. A. 326
Il'in, V. A. 288
Il'yušin, A. A. 353
Ioanin, G. 375
Ishiguro, E. See Kotani, M.
Ishiguro, E.-Arai, T.-Sakamoto, M. 324
Ishiguro, K. 268
Isiwata, T. 298
Islinskiĭ, A. Yu. 339
Ivanenko, D. D. See Sokolov, A. A.
Ivanov, V. K. 261
Iwata, K. 302
Jacobs, K. 295
Jacobsen, B. B. 374
Jacobson, N. 247
Jakšić, B. See Glaser, V.
Jans, J. P. 249
Jans, J. P.-Nakayama, T. 250
Jaynes, E. T. 335
John, F. 323
Johnson, S. M.-See Bradt, R. N.
de Jongh, B. H. 373
Jongmans, F. 315
Joshi, D. D. See Adhikari, B. P.
Jost, R. 277
Judd, B. R. See Elliott, J. P.
Judd, D. L. See Rosenbluth, M. N.
Kaempffer, F. A.-Kennedy, E. M. 365
*Kagan, V. F. 303
Kahana, S.-Polkinghorne, J. C.-Martin, J. L. 362
Kahane, J.-P.-Salem, R. 268
Kalmár, L. 237
Kämmerer, W. 325
*Kanger, S. 239
Kanzaki, H. 344, 345
Karlin, S. See Bradt, R. N.
Karrass, A.-Solitar, D. 248
Kato, T. 279
Kawakami, Y. 262
Každan, Ya. M. See Žukov, A. I.
Keats, J. A. 329
Keller, G. 369
Keller, J. B. 348
Kempthorne, O. See Wilk, M. B.
Kendall, D. G.-Moyal, J. E. 295
Kennedy, E. M. See Kaempffer, F. A.
Kennedy, P. B. 265
Kerner, E. H. 374
Kestelman, H. 269
Khan, N. A. 242
Khintchine, A. 328
Kiefer, J.-Weiss, L. 333
Kil'čevskiĭ, N. A. 341
Kimura, T. See Kotani, M.
Kinukawa, M. 269
Kirby, D. 319
Kiselev, A. A. 348
Kitagawa, T. 330
Kitagawa, T.-Seguchi, T. 333
Kleene, S. C. 238
Klein, B. 339
Klein, L. R. 372
Kletter, L. 371
Klimov, N. I. 251
Knowles, J. K.-Reissner, E. ... 340
Kobayashi, S. 313
Koch, R. J. 290
Kočina, I. N. 347
Kogan, S. Ya. 338
Kolibiar, M. 243
Kolodner, I. I. 285
Korányi, A. See Sz.-Nagy, B.
Košelev, A. I. 286
Kosticyn, V. T. 334
Kostomarov, D. P. 272
Koszul, J. L. 301
*Kotani, M.-Amemiya, A.-Ishiguro, E.-Kimura, T. 324
Kotzig, A. 240
Krabbe, G. L. 294
Kramer, C. Y. 331
*Kramers, H. A. 361
Krein, M. G. 277
Kreyszig, E. 282
Krishna Iyer, P. V.-Bhattacharyya, M. N. 329
Kručkovič, G. I. 312
Kubiński, T. 239
Kudryavcev, L. D. 283
Kuerti, G.-Faymon, K. 353
Kulik, S. 257
Kumari, Sulaxana. 269
Kummerer, K. 358
Kunle, H. 308
Kuo, Y. H. 352
Ladik, J.-Csukás, A. 365
Ladyženskaya, O. A. 279
Lakshmana Rao, S. K. 348
Lamprecht, E. 244, 245
Landsberg, P. T. See Elcock, E. W.
Lang, S. 320
Lang, S.-Serre, J.-P. 320
Laptev, B. L. 314
Lasota, A. 284
Laurent, A. G. 333
Lavrent'ev, M. A.-Šabat, B. V. . 286
Lax, P. D. 281
Lebedev, N. N. See Grinberg, G. A.
Lee, L. H. N.-Ades, C. S. 342
Legendre, R. 351
Leimbacher, W. 373
Lelek, A. 300
Lemmer, R. H. See Frahn, W. E.
Lense, J. 257
Lenz, F.-Scheffels, W. 356
Lenz, H. 263
Leptin, H. 245, 290
Leruste, P. 334
Levallois, J.-J. 370
Leżański, T. 297
Li, J. C. M. See Ting, Tsuan Wu.
Lighthill, M. J. 351, 352
Lindley, D. V. 330, 335
Lingenberg, W. 311
Linnik, Yu. V. See Zinger, A. A.
Lions, J. L. See Delsarte, J.
Loinger, A. See Bocchieri, P.
Longuet-Higgins, M. S. 328, 329
Lorent, H. 303
Louhivaara, I. S. 281
Low, F. E. 366
*Luce, R. D.-Raiffa, H. 373
Lund, I. A. See Gringorten, I. I.
Lunts, A. G. 375
Lupanov, O. B. 360
Łuszczewska-Romahnowa, S. .. 237
McCoy, N. H. 246
MacDonald, W. M. See Rosenbluth, M. N.
MacDonald, W. M.-Rosenbluth, M. N.-Chuck, Wong. 335
McDowell, E. L. See Sternberg, E.
McKean, H. P., Jr. 285
McKean, H. P., Jr. See Feller, W.
McLeod, R. M.-Gergen, J. J.-Dressel, F. G. 281
McNaughton, R. 240
McVittie, G. C. 370
*McVittie, G. C. 370
Maass, H. 252, 264
Macbeath, A. M. 255
Machol, R. E. See Goode, H. H.
Magenes, E. 262
Mal'cev, A. I. 240
Malgrange, B. 280
Malkus, W. V. R. 350
Mallows, C. L. 326, 330
*Mandl, F. 361
Mangler, K. W. 352
Marčenko, V. A. See Agranovič, Z. S.
Mařík, J. 256
Markov, A. A. 240

AUTHOR INDEX

(Continued from cover 3)

Markus, S. 254
Martić, L. 243
Martin, A. V. 255
Martin, J. L. See Kahana, S.
Marx, G. 365
Mascarenhas, S. 360
Matsushima, Y. 315
Maue, A.-W. See Höni, H.
Mayer, O. 311
Merryman, J. D. See Halstead, M. H.
Meschkowski, H. 238
Mestel, L.-Spitzer, L., Jr. 369
Miasek, M. 346
Michael, E. 299
Michael, J. H. 300
Michel, L. 366
Mientka, W. See Hawkins, D.
Mihailov, Yu. A. 360
Mihailović, D. 368
Mikhail, M. N. 259
*Miller, K. S. 258
Miller, M. A. See Gringorten, I. I.
Miller, M. A.-Talanov, V. I. 359
Mindlin, R. D. See Deresiewicz, H.
Mirels, H.-Braun, W. H. 349
Miron, R. 306
Mirzadžanzade, A. H.-Abbasov, A. A. 344
Mishra, R. S.-Upadhyay, M. D. 306
Mizohata, S. 285
Moiseev, N. N. 348
Moisil, G. C. 375, 376
Moisil, G. C.-Nedelcu, M. 375
Moisil, G. C.-Popovici, C. P. ... 375
Mokriščev, K. K. 305
Montaldi, E. See Gulmanelli, P.
Montgomery, D.-Yang, C. T. .. 291
Moodie, A. F. See Cowley, J. M.
Moore, P. G. 330
Morita, K.-Hanai, S. 299
Morris, C. R. 358
Moser, J. 278
Moser L.-Wyman, M. 268
Mostow, G. D. 291, 292
Mostowski, A. 237, 238, 240
Motzkin, T. S.-Taussky, O. 242
Moyal, J. E. See Kendall, D. G.
Mrówka, S. 299
Murota, T.-Ueda, A. 363
Murota, T.-Ueda, A.-Tanaka, H. 363
Musielak, J. 269
Muster, D. F.-Sadowsky, M. A. 340
Myrberg, P. J. 263
Nagata, J. 300
Naghdi, P. M. 339, 342
Nakamura, M. 286
Nakanishi, S. 256
Nakayama, T. See Jans, J. P.
Nardini, R. 354
Nash, W. A. 341
Nasr, S. K. 325
Nedelcu, M. 257
Nedelcu, M. See Moisil, G. C.
Neidhardt, G. L.-Sternberg, E. 338
Néron, A. 321
Neugebauer, C. J. 256
Nicholson, A. N. 250
Nicholson, W. L. 326
Nickel, K. 347
Ninomiya, N. 262
Niven, I.-Warren, L. J. 246
Noilet, L. 319
Nonweiler, T. 322
Norman, N. 344
Nottrot, R. 304
Nowacki, W.-Sokołowski, M. .. 342
Nožička, F. 311
Numakura, K. 290
Oakley, C. O.-Wisner, R. J. 241
Obreschkoff, N. 266
Oikawa, K. 258
Okubo, T. 313
Olech, C. 271
Opial, Z. 271
Orlov, S. A. 277
Osserman, R. 288
Ostrovskiĭ, G. M. 324
Otrokov, N. F. 278
Ōtsuki, T. 314
Ozawa, M. 259
Pai, Shih-I. 354
Palamà, G. 251
Paria, G. 337
Parkus, H. 340
Parodi, M. 242
Peck, J. E. L.-Dulmage, A. L. 374
Perfect, H. 241
Pesin, I. N. 258
Petermann, A. See Steuckelberg, E. C. G.
Petržak, K. A. See Bak, M. A.
Pfouts, R. W. 371
Philip, J. R. 354
*Pi Calleja, P. 241
Picht, J. 356
Piddington, J. H. 369
Pisareva, N. M. 312
Piszczek, K. 341
Pitt, H. R. 267
du Plessis, N. 270, 271
Plunkett, R. L. 301
Pogorelov, A. V. 309
*Pogorelow, A. W. 305
Pogorzelski, W. 282
Poirier, Y. See Crausse, E
Polkinghorne, J. C. See Kahana, S.
Popescu, I. 311
Popov, E. P. 375
Popovici, C. P. See Moisil, G. C.
Postnikov, A. G. 252
Povarov, G. N. 360, 375
Power, E. A.-Zienau, S. 365
Preston, J. H. 347
Prokopovič, I. E. 342
Pursell, L. E. 297
Raiffa, H. See Luce, R. D.
Rashevsky, N. 374
Rasiowa, H.-Sikorski, R. 240
Rauch, H. E. 261
Rayski, J. 366
Rayski, J. See Hanus, W.
Ree, Rimhak. 247
Régnier, A. 325
Rehlicki, Z. I. 275
Reid, W. T. 275
*Reidemeister, K. 237
Reissner, E. See Knowles, J. K.
Reiter, H. 290
Rembs, E. 308
Remmert, R. See Grauert, H.
Reuter, H. 371
Ricci, G. 257
Richardson, M. 237
Richman, R. L. See Halstead, M. H.
Richter, E. 354
Rim, Dock Sang. 244
*Risser, R.-Traynard, C.-E. . 329
Roberts, J. B. 250, 265
Rogers, M. H. 353
Röhrl, H. 274
Romanov, Yu. F. See Bak, M. A.
Rooney, P. G. 270
Rosculet, M. N. 246
Rose, A. 239
Rosenauer, N. 334
Rosenbluth, M. N. See MacDonald, W. M.
Rosenbluth, M. N.-MacDonald, W. M.-Judd, D. L. 335
Rosenlicht, M. 321
Rosenstock, H. B. 254
Roussopoulos, P. 361
Roussopoulos, P.-Vardalakis, G. 362
Rüdiger, D. 340
Rumney, M. 250
Runciman, W. A. See Elliott, J. P.
Rusak, B. I. 282
Ryabova, E. V. 342
Šabat, B. V. See Lavrent'ev, M. A.
Sadowsky, M. A. See Muster, D. F.
Sakamoto, M. See Ishiguro, E.
Salem, R. See Kahane, J.-P.
Samokiš, B. A. 322
Sanoyan, V. G. 346
Sapondžyan, O. M. 339
Sarhan, A. E.-Greenberg, B. G. 331
Sasayama, H. 314
Sauter, E. See Braunbek, W.
Savage, I. R. 331
Schäfke, F. W. 264, 296
Scheffels, W. See Lenz, F.
Scheffler, H. 369
Scheibe E. 313
Schieferdecker, E. 295
*Schneider, T. 252
Schützenberger, M. P. 249
*Sedov, L. I. 346
Seebeck, C. L., Jr.-Hoelzer, H. . 324
Segre, B. 319
Seguchi, T. See Kitagawa, T.
*Séminaire Schwartz de la Faculté des Sciences de Paris, 1954/1955. 279
Sengupta, A. M. 337
Serre, J.-P. 320
Serre, J.-P. See Lang, S.
Shapovalov, V. P. 278
Shen, Chi-Neng. 279
Shephard, G. C. 248
Siebertz, K. See Groschwitz, E.
Siegel, K. M. 264
Siforov, V. I. 374
Šik, F. 298
Sikorski, R. See Rasiowa, H.
Šil'krut, D. I. 284
Simon, A. B. 302
Sinclair, A. 298
Siotani, M. 331
Siryk, G. V. 258
Skalskaia, I. P. See Grinberg, G. A.
Sklianski, A. L. 368
Ślebodziński, W. 313
*Slězkin, N. A. 349
Słowikowski, W. 299
Smale, S. 302
Smiley, M. F. 245
Smirnov, Yu. M. 300
Smith, A. M. O. 349
Smith, F. B. 350
Smith, K. T. 261
Snell, J. L. See Berger, J.
Sokolov, A. A.-Tsytovich, V. N. 363
Sokolov, A. A.-Ivanenko, D. D.-Ternov, I. M. 362
Sokolovskiĭ, V. V. 343
Sokołowski, M. 342
Sokołowski, M. See Nowacki, W.
Solitar, D. See Karrass, A.
Sörensen, W.-Bader, R. 259
Spampinato, N. 318
Spanier, E. 317
Spar, J. 371
*Specht, W. 250
Spencer, D. C. See Gugenheim, V. K. A. M.
Spitzer, L., Jr. See Mestel, L.
Spurny, H. See Grümm, H.
van der Spuy, E. 364
Stečkin, S. B. See Zuhovickiĭ, S. I.
Steffensen. J. F. 369
Stein, E. 318
Sternberg, E. See Neidhardt, G. L.
Sternberg, E.-McDowell, E. L. 343
Steuckelberg, E. C. G.-Petermann, A. 363
Stewartson, K. 354
Stiefel, E. 324
Stolt, B. 251
Stone, A. H. 299
Stout, T. M. 375
Straszewicz, S. 304
Straus, A. V. 277
Štrel'cov, V. V. 307
Strubecker, K. 311, 312
Stummel, F. 283
Stuvel, G. 371
Šulikovskiĭ, V. I. 307
Šul'man, T. A. 307
Sultanov, G. F. 368
Summarized proceedings of a conference on contemporary optics 356
Sumner, F. H. See Brooker, R. A.
Suzuki, M. 248
Swineford, F.-Fan, Chung-Teh. 333
Szász, P. 304
Sz.-Nagy, B. 296
Sz.-Nagy, B.-Korányi, A. 296
Tabueva, V. A. 278
Takashima, M. 327
Takeno, H. 367
Takeuti, G. 237
Talanov, V. I. See Miller, M. A.
Tallini, G. 316
Tamura, T. 243
Tanaka, H. See Murota, T.
Tashiro, Y. 316
Taucer, G. 273
Taussky, O. See Motzkira T.
Taylor, R. L. 289
Taylor, S. J. 255
Tedeschi, B. 326
Terehova, N. P. 299
Ternov, I. M. See Sokolov, A. A.
Thom, R. 310
Thorne, R. C. 272
*Thrall, R. M.-Tornheim, L. .. 241
Ting, Tsuan Wu.-Li, J. C. M. .. 344
Ting Kuan, Shu-chang. 325
Tits, J. 269
Tobocman, W. 364
Todeschini, B. 352
Tollmien, W. 360
Tornheim, L. See Thrall, R. M.
Trakhtenbrot, B. A. 340
Traynard, C.-E. See Risser, R.
Tretner, W. 356
Trifan, D. 342
Trohimčuk, Yu. Yu. 257
Tschobanow, I. See Dolaptschiew, B.
Tsian, Tse-pei. 327
Tsytovich, V. N. See Sokolov, A. A.
Tukey, J. See Cornfield, J.
Ueda, A. See Murota, T.
Uflyand, Ya. S. See Grinberg, G. A.
Umegaki, H. 294
Unger, H. 323
Upadhyay, M. D. See Mishra, R. S.
Upton, C. J. F. 269
Urban, A. 306
Urbanik, K. 326, 376
Utz, W. R. See Brunk, H. D.
Vaccaro, M. 303
Vaida, D. 376
Vardalakis, G. See Roussopoulos, P.
Varga, O. 312
Varoli, G. 322
Vasilache, S. 273
Vekua, I. N. 308
Vertgeim, B. A. 267
*Villaseñor, Z. F. 251
Volosov, V. M. 276
Vol'vič, S. I. 341
Vučković, V. 265
Wada, T. 302
Walker, A. G. 312
Wallace, A. H. 320
Wallisch, W. 341
Walsh, J. E. 330
Walsh, J. L. See Fekete, M.
Walter, W. 262, 283
Warburton, G. B. 341
Warren, L. J. See Niven, I.
Watanabe, Y. 270
Ważewski, T. 272
Weber, J. 346, 347
Weier, J. 302
Weinert, H. J. 251
Weisner, L. 264
Weiss, L. See Kiefer, J.
Weisskopf, V. F. 364
Wentzel, G. 366
Whitehead, J. H. C. 300
*Whitney, H. 309
Wigner, E. P. See Hewitt, E.
Wilcox, C. H. 358
Wilder, R. L. 300, 303
Wilk, M. B.-Kempthorne, O. ... 332
Wilkens, A. 368
Willis, B. T. M. 344
Winogradzki, J. 362
Winther, A. See Alder, K.
Wintner, A. 271, 284, 326
Wiskott, D. 357
Wisner, R. J. See Oakley, C. O.
*Wolf, P. 247
Wolk, E. S. 243
Wong, D. Y. 364
Wyman, M. See Moser, L.
Wymore, A. W. See Hammer, P. C.
Wynn, P. 265
Yang, Chung-Tao. 292
Yang, Chung-Tao. See Montgomery, D.
Yano, S. 269
Yeh, Hsuan. 351
Yen, Ti. 246
*Yosida, K. 294
Yu, Chia-Yung. 265
Yûjôbô, Z. 258
Yukawa, H. 366
Zadiraka, K. V. 272
Zaharov, D. A. 307
Žautykov, O. A. 286
Zemanian, A. H. 270
Zienau, S. See Power, E. A.
Žigulev, V. N. 352
Zinger, A, A.-Linnik, Yu. V. ... 325
Zuber, R. 323
Zubieta, R. G. 240
*Zubov, V. I. 275
Zuckerman, H. S. See Hewitt, E.
Zuhovickiĭ, S. I.-Stečkin, S. B. 267
Žukov, A. I.-Každan, Ya. M. ... 353
Zykov, A. A. 246